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ABSTRACT

The study was designed to describe the basis of a potential mathematics learning theory founded on the relationship between language and thinking, to relate the subjects' overt verbalization and performance after they had been taught a mathematical structure, to relate overt verbalization and the number of discovered rules, and to examine the strategies used. Forty girls, 11-12 years old, were randomly assigned to one of four instructional situations: (1) subjects talk aloud while doing mathematical activities and then are silent, (2) subjects are silent while doing the activities and afterwards answer questions and explain findings, (3) subjects verbalize both during and after the activities, and (4) subjects do not verbalize either during or after the activities. A machine wired to embody the Klein-Four Group structure was used as a manipulative aid. Among the results found were: (1) subjects performed better, retained more and discovered more rules when they were silent while doing the activities and afterwards answered questions and explained findings; (2) there was significant interaction between learning and questioning; (3) the subjects' overt verbalization during questioning did not accelerate learning; and (4) the nature of the rule and the number of discovered rules influenced performance. (Author/DT)

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Technical Report No. 293

THE RELATIONSHIP BETWEEN GIRLS' OVERT VERBALIZATION, PERFORMANCE, RETENTION, RULES AND STRATEGIES AS THEY LEARN A MATHEMATICAL STRUCTURE: A STUDY BASED ON ELEMENTS OF A POTENTIAL THEORY WHICH RELATES THINKING, LANGUAGE AND LEARNING

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Principal Investigator

Report from the Project on Conditions of
School Learning and Instructional Strategies

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STATEMENT OF FOCUS

Individually Guided Education (IGE) is a new comprehensive system of elementary education. The following components of the IGE system are in varying stages of development and implementation: a new organization for instruction and related administrative arrangements; a model of instructional programing for the individual student; and curriculum components in prereading, reading, mathematics, motivation, and environmental education. The development of other curriculum components, of a system for managing instruction by computer, and of instructional strategies is needed to complete the system. Continuing programmatic research is required to provide a sound knowledge base for the components under development and for improved second generation components. Finally, systematic implementation is essential so that the products will function properly in the IGE schools.

The Center plans and carries out the research, development, and implementation components of its IGE program in this sequence: (1) identify the needs and delimit the component problem area; (2) assess the possible constraints--financial resources and availability of staff; (3) formulate general plans and specific procedures for solving the problems; (4) secure and allocate human and material resources to carry out the plans; (5) provide for effective communication among personnel and efficient management of activities and resources; and (6) evaluate the effectiveness of each activity and its contribution to the total program and correct any difficulties through feedback mechanisms and appropriate management techniques.

A self-renewing system of elementary education is projected in each participating elementary school, i.e., one which is less dependent on external sources for direction and is more responsive to the needs of the children attending each particular school. In the IGE schools, Center-developed and other curriculum products compatible with the Center's instructional programing model will lead to higher morale and job satisfaction among educational personnel. Each developmental product makes its unique contribution to IGE as it is implemented in the schools. The various research components add to the knowledge of Center practitioners, developers, and theorists.

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ABSTRACT

The study was designed to describe the basis of a potential mathematics learning theory founded on the relationship between language and thinking; to gather evidence to test the validity of an initial empirical proposition relating to the subjects' overt verbalization and performance after they had been taught a mathematical structure; to gather evidence to test the validity of a second empirical proposition relating to the subjects' overt verbalization and the number of discovered rules, which define the mathematical structure; and to examine the strategies the subjects used in the learning of those rules.

Forty girls, 11-12 years old, from the Verona Elementary School were randomly assigned and individually asked to participate in one of the four instructional situations: (a) subjects talk aloud while doing mathematical activities and then are silent after the activities ($L\bar{Q}$), (b) subjects are silent while doing the activities and afterwards answer questions and explain findings ($\bar{L}Q$), (c) subjects verbalize both during and after they have done the activities (LQ), and (d) subjects do not verbalize either during or after they have done the activities ($\bar{L}\bar{Q}$). The subjects used a machine that was wired to embody a group structure, in this case, Klein's Four Group. The actual stimuli were pairs of four geometric figures: a circle, a

square, a star and a triangle, lighted by lamps. By manipulating the machine, the subject gradually discovered the rules of the group.

The results of the study indicate that (1) subjects performed better, retained more and discovered more rules when they were silent while doing the physical mathematical activities and afterwards answered questions and explained findings; (2) there was significant interaction between learning and questioning; (3) the subjects' overt verbalization during questioning did not accelerate the learning process, which implies that language does not act as a shifter from the planes of perception and action to the plane of mental representation under these circumstances; (4) the subjects should work without overt verbalization during physical mathematical learning activities; (5) the nature of the rule to be discovered influences performance; (6) the number of discovered rules influences performance; (7) identity was the most difficult rule for subjects to discover; (8) the same subject used different strategies for different rules, but the great majority who discovered rules used the synthetical strategy, the implication being that when a subject discovers a rule, he tries to check it; (9) the consecutiveness method of deriving rules is appropriate for observing the strategies used by the subjects; and (10) the subjects who discovered the identity rule used the synthetical strategy.

Chapter I

INTRODUCTION

Statement of the Problem

The objective of this study was fourfold: (1) to describe the basis of a potential mathematics learning theory founded on the relationship between language and thinking, (2) to gather evidence to test the validity of an initial empirical proposition relating subjects' overt verbalization to performance after they had been taught a mathematical structure, (3) to gather evidence to test the validity of a second empirical proposition relating subjects' overt verbalization to the number of discovered rules which define the mathematical structure, and (4) to examine the strategies the subjects used in the learning of these rules.

The study is theory-oriented since its hypotheses come from a potential theory. The theory which relates language, thinking, universals, analysis, synthesis, and motivation is presented in Chapter II. If the theory has value, it will first express propositions and second, if validated, resolve some important problems of mathematics learning.

The phenomenon of overt verbalization was chosen since it includes the four following conditions: (1) embodies language and thinking, (2) can be manipulated in an experimental setting, (3) is common in the teaching of mathematics, and (4) provides a bridge for studying learning processes.

Method and Brief Description of the Experiment

The method used consisted of presenting the subject with pairs of geometric figures on a machine with the same figures lighted by lamps. The machine was wired to embody a group structure, in this case, Klein's Four Group. The actual stimuli were pairs of four geometric figures: a circle, a square, a star, and a triangle. By manipulating the machine, the subject discovers that the lighted figures refer to definite combinations of pairs of figures.

One value of the method is that the procedure used by the subject can be analyzed in detail. The subject was free to choose the pair of figures and to predict the result on answer sheets by marking his choices and predictions. The answer sheet represented the same pairs of figures as the machine. The main task for the subject was to discover the rules which control the functioning of the machine. His sequence of choices and predictions reflects nearly every step of his reasoning: forming hypotheses, testing them, and discarding them. Thus, the experiment was not only a test of the presence of rules or the use of strategies,

but also an investigation of the learning of mathematical structures. Terminal performance and retention were measured using the same test which involved having subjects answer each of the possible 48 open sentences for the Klein Four Group.

The population used in this study was girls aged 11 and 12. The choice of this group was based on information gathered in a previous study (Pereira & Romberg, in press), which indicated that only girls at least 11 years old were able to discover any rules.

Significance of the Study

The significance of overt verbalization, mathematical structures and strategies, in the context of mathematics education is also important to this study. The importance of overt verbalization to learning has been the concern of researchers in different fields such as psycholinguistics and psychology. However, the findings in those areas raise numerous contradictions, inconsistencies, and disagreements. Often the research cannot be generalized to encompass the learning of mathematics. As a result, some mathematics educators have become proponents of verbalization, while others maintain that learning to verbalize a mathematical experience adds nothing to one's understanding of a concept, or even that it interferes with the ability to apply the concept. If theories of overt verbalization are to be applied effectively to mathematical learning, mathematics educators will have to conduct their own research.

Hendrix and Ausubel give the main characteristic arguments of the divergent groups of educators about overt verbalization. Hendrix, "Learning by Discovery" (1961), on one side of the issue, advocates that the teacher delay verbalization of discovered generalizations on two grounds: (1) that there is evidence that a student does not have the linguistic capacity to state his discovery with precision, and that imprecise verbalization may have undesirable effects, and (2) that there is research showing that a student who immediately attempts to state his discovery is less able to use that discovery than one who possesses the discovery on a nonverbal awareness level.

Ausubel (1968), on the other hand, argues that the verbalization of a subverbal insight is an integral part of the thinking process, suggesting that leaving a discovery at a nonverbal level would actually abort the thinking process. He criticizes Hendrix's view that verbalizing an insight before it is used may interfere with its transfer to other situations as illogical and lacking in empirical support. He does concede, however, that attempting to verbalize a nonverbal insight before the principle is clearly understood interferes with its transferability. The propositions posed in this study, if validated, should clarify this problem.

The activities performed by the children in this study were based on mathematical structures. These structures were used because they are important in the construction of mathematics, in mathematics models for measurement, in cognitive processes,

and in mathematics education. The mathematician, Glivenko (1938), explains that mathematicians have shown that Set Theory, Group Theory, Number Theory, Projective Geometry, Topology, Probability, Logic, and Quantum Mechanics have the same formal properties, which persist from the elementary notions to the most abstract concepts of those subjects. The relations are called structures, and they yield the possibility of a better understanding and unification of mathematics. Emphasis on mathematical structures is becoming more and more widespread in mathematics because they unify parts of different fields. Frequently they allow for clarification and provide the impetus for new developments, even though they have no necessary logical connection with the physical world. The mathematician, Horst Schubert (1972), observes:

> At the present time, axiomatic theories play an important role in mathematics. In them one considers sets with a given structure, the "Models of the theory" (e.g. real vector spaces, group, or topological spaces) and structure preserving maps between them (e.g. linear maps, homomorphisms) (Schubert, 1972, p. V).

The importance of mathematical structures in relation to cognitive processes has been pointed out by the psychologist, Jean Piaget (1955). He raised the following questions: (1) Are mathematical structures simply an artificial product of a theoretical and axiomatic analysis, or are they natural? and (2) Do mathematical structures correspond to something in human intelligence, particularly in the intelligence of children? Piaget concludes from his research (1964) that mathematical

structures do correspond to something in natural intelligence and are developed from childhood through adolescence.

Recommendations of various groups considering mathematics education at levels from kindergarten through college have emphasized the early introduction of the concept of structure in mathematics. Indeed, says Taylor (1965), "If we were to select one emphasis that is characteristic of the new curricula in mathematics at all levels, that emphasis would likely be the emphasis on structure in mathematics" (Taylor, 1965, p. 226). Adler (1971) comments that "the modern ideas of mathematical structure make learning easier because they simplify and unify what the children have to learn" (Adler, 1971, p. 70). He continues:

> Mathematics was taught in the past as a jumble of disconnected facts. But if we introduce at an early stage the advanced ideas of mathematical structure, all the formerly disconnected mathematical facts will fall into place as part of a coherent whole. They will begin to make sense, and mathematics will be an easier subject to learn (Adler, 1971, p. 70).

The ideas of Dienes (1971) on mathematical structures are especially interesting. He states:

> Mathematics is the study of structures. We are not born knowing these structures. So it is a relevant question to ask how these structures become built up. At first glance, it is painfully obvious that such structures must in the last resort be built up out of our experiences as we cope with our environment. These experiences may possibly be either secondhand or imaginary; during the part of childhood described by Piaget as the concrete operational stage - i.e., between ages seven and eleven, or thereabouts - the most effective learning of a

> permanent nature takes place as a result of concrete, not hypothetical, experiences. Once mathematical structures have been built up, it is possible to wonder whether we are capable of sorting out the relationships between these structures or the internal relationships within each structure (Dienes, 1971, p. 222).

The emphasis given by mathematicians to mathematical structures supports the opinion of educators that the subject should be taught from elementary to advanced levels using the same unifying concepts and language. This emphasis is consistent with the recommendation of those educators and psychologists who maintain that both understanding of concepts and the potential for transfer of knowledge are acquired through mastery of underlying principles and relationships.

Group structure, the specific structure used in this study, is one of the most important mathematical structures. Barbeau (1968) comments:

> Various reasons support the selection of the group as an appropriate unit of structure for study at the secondary level. The frequent appearance of group properties in systems which are externally unrelated makes the concept of group an interesting idea of wide application. The group concept has importance in itself, since the significance and usefulness of many of the classical mathematical structures are consequences of their group properties. Moreover, the study of groups can serve as a convenient vehicle for the introduction of deductive reasoning because of the simplicity of the axiom system by which a group is defined (Barbeau, 1968, p. vi).

The strategies used by subjects in problem solving or in the acquisition of a concept are also investigated in this study. Strategies have rarely been studied in connection with the learning of school subjects, but they have been explored in laboratory

situations. Individuals may learn to instruct themselves and to adopt strategies which guide their thinking when engaged in problem solving. Bruner, Goodnow, and Austin (1956) define the objectives of a strategy as follows:

> a. To insure that the concept will be attained after the minimum number of encounters with relevant instances.
>
> b. To insure that a concept will be attained with certainty, regardless of the number of instances one must test <u>en route</u> to attainment.
>
> c. To minimize the amount of strain on inference and memory capacity while at the same time insuring that a concept will be attained.
>
> d. To minimize the number of wrong categorizations prior to attaining a concept (Bruner, Goodnow & Austin, 1956, p. 54).

Branca and Kilpatrick (1972) remark that the term "strategy" abounds these days in our educational literature. They state:

> Researchers in mathematics education should be cautious in using the term. One assumes, for example, that if a person has a strategy for playing a game, he will be aware that he is acting according to some plan, however vague or rudimentary. If the pattern of questions a person asks or moves he makes is to be called a strategy, moreover, then one ought to expect that when faced with a similar task on another occasion, the person's behavior will show a similar pattern (Branca & Kilpatrick, 1972, p. 132).

On the same issue, Bruner, Goodnow and Austin (1956) give the following comment:

> A strategy as we are using the term does not refer to a conscious plan for achieving and utilizing information. The question whether a person is or is not conscious of his strategy, while interesting, is basically irrelevant to our inquiry. Rather, a strategy is inferred from the pattern of decisions one observes in a problem-solver seeking to attain a concept (Bruner, Goodnow & Austin, 1956, p. 55).

The importance of strategies for problem solving in education is explicit in the words of Gagné (1970), who says:

> Obviously, strategies are important for problem solving, regardless of the content of the problem. The suggestion from some writings is that they are of overriding importance as a goal of education (Gagné, 1970, p. 232).

With the above considerations as background, the following questions can be raised: (1) What is language?, (2) What are the characteristics of language?, (3) What is thinking?, (4) What is the relationship between thinking and language?, (5) What is the relationship between language and learning?, (6) What is overt verbalization?, (7) What is a mathematical structure?, (8) What is a group?, (9) What is a Klein group?, and (10) What is a strategy? An attempt is made in Chapter II to answer each of these questions by describing the basis of a potential learning theory for mathematics.

Chapter II

THEORETICAL BACKGROUND

Origin of the Potential Theory

When the author started to teach mathematics in Brazil in 1954, he was influenced by the magister dixit of the time. The ideas of Comenius, Pestalozzi, Herbart, Froebel, Montessori, Decroly, Thorndike, Dewey, Young, Adam, and others were in fashion. The teacher was the focal point of a class. Teaching success was determined by the way instructional techniques were used, by the individual's mastery of subject matter, and by his general cultural background. Teachers used sophisticated techniques in the organization of the immediate and mediate objectives of a course and in the preparation of their lectures. Like actors, teachers performed using a blackboard, a flannel-board, techniques of computation with an abacus, a slide projector, a wallboard, geometric solids, machines for demonstrating the theorems of geometry (Thales, Pythagoras, etc.), as well as a great number of specific formulas for teaching arithmetic, algebra, geometry, or trigonometry. Such questions as: What does one teach?, When does one teach?, How does one teach?, and Why does one teach?, when not answered in the official

curricula, were left to the teacher's discretion. Yet, in spite of the well prepared teacher's plan of lectures and his skillful utilization of audio-visual aids, the children did not like mathematics. The formulas could not solve the problems of teaching mathematics.

By 1957 the author's views started to change after reading the works of Polya, Hadamard, Gattegno, Cuisenaire, Gastelnuevo, Choquet, Aebli, and the researchers of Piaget's group. At that time the author wrote, "New Ways in the Teaching of Mathematics" (Pereira, 1957). This work emphasized the discovery method and the utilization of the dynamic materials. The child's actions, not the teacher's, were the focus of attention. Thus, the old questions were reformulated in relationship with the children's problems. As a result of this publication, a dialogue was started with members of the reform movement in mathematics in Brazil. Congresses, seminars, and national meetings gave the author the opportunity to establish a new teaching philosophy in Brazil.

In 1958 the author wrote, "The Resolution of Elementary Mathematical Problems," which was influenced by the works of Dewey, Polya, Claperède, and Wertheimer. That publication stressed the following topics: (1) the nature of the mathematical problems, (2) the functions of mathematical problems, (3) the characteristics of the solver, (4) logical analysis and psycho-pedagogical research, (5) methods of resolution, and (6) general

suggestions for the teaching and the techniques of problem solving. Two of the conclusions in that work mentioned "language":

> It is important to study mathematics as if it were a language. Exercises which involve translation contribute to an increase in the capability of problem solving and provide the motivation to study mathematics (Pereira, 1958, p. 78).

and

> It is very important to do research whose object is to identify conditions for organizing a more appropriate curriculum (Pereira, 1958, p. 78).

Because problem solving became a main concern, the author was invited by the Ministry of Education and by different Brazilian institutions to conduct inservice programs for teachers of the elementary and secondary levels. The Catholic University of Pernambuco invited the author to be Professor of Special Didactics of Mathematics. This experience made increasingly apparent the need for a general mathematics learning theory capable of predicting and explaining student behavior. The time had come to eliminate the alchemist influence in education and to avoid magic recipes.

In 1959 the author was invited to Paris by the Embassy of France in Brazil to participate in a program of studies in the "Centre International d'Études Pédagogiques" of the University of France. This afforded the opportunity to learn about the organization of the educational system of France and to participate in many activities relating to research and theoretical issues. A visit to the schools of Belgium was arranged in order to learn more about such dynamic materials as Cuisenaire rods, algeblocs, geospaces,

geoplanes, etc. Also in Louvain, Belgium, the author had the opportunity to visit the Laboratory of Experimental Psychology, administered by the Buyse's group and devoted primarily to research in problem solving.

Back in Brazil, the author wrote "Dynamic Mathematics with Numbers in Colors" (Pereira, 1961) and organized two mathematical expositions of dynamic materials. In the last chapter of that book, the relationship between language and problem solving is again stressed.

Then "Modern Course of Mathematics" (Pereira, 1962), which emphasizes mathematics as a language, was published. The exercises in this work ask students to translate problems from a verbal form to a symbolic form and vice versa. Also, a dictionary was compiled which explains the meaning of mathematical terms and expressions. Students were expected to invent new algorithms which were then to be used as exercises for others. The successful performance of students utilizing this program led the author to the first hypothesis--the study of mathematics as a language, that is, the mastery of the signs and the relations and operations of mathematics, increases the scope and quality of a child's mathematical creativity. At that time, the questions, Is mathematics really a language?, What differences exist among languages?, and Is mathematics a special language or the same language used to express everyday thoughts?, became paramount.

In 1965, the author was invited to work in the Teaching Center of Sciences of the Northwest of the Federal University of Pernambuco. And in 1966, the author became Professor of Mathematics of the Master Program for psychologists, sociologists, and economists of the Federal University of Pernambuco. In this capacity the author was able to learn more about psychology and sociology.

In 1971, the author enrolled in a Graduate Program in Mathematics Education at the University of Wisconsin-Madison with support from the Ford Foundation. This opportunity opened to the author a new world of readings and ideas about the relationship between language and mathematics. The author became more convinced about the possibility of building a mathematical learning theory based on language. In particular, studying the works of Piaget, Vygotsky, Bruner, Chomsky, Roger Brown, Lenneberg, Saussure, Sokolov, Skinner, Sapir, Miller, Carroll, Fodor, Slobin, Pribram, Shannon, Cherry, Bridgman, Whorf, and others under the direction and criticism of Professor Romberg sharpened emerging notions. In the summer of 1972, the author participated in a research study which analyzed the speech behavior of children. The study was planned and directed by Professor Romberg (Romberg, Jurdak, Pereira, & Green, in press). This experience made the importance of the relationship between language and thinking more apparent.

From these experiences the elements of a potential theory presented in the next section began to take shape. But prior to the formulation of a theory, the following questions had to be answered: What is thinking?, What is language?, Do thinking and

language have the same origin?, What is the relationship between language and thinking?, and What is the relationship between language and mathematics learning? The answers presented below do not belong to anv particular scholar; rather they are the synthesis of the author's experience and reading. They explain the relationships between language and thinking in a way new to the author and can serve as the base for the construction of a mathematical learning theory. The details of the theory in its present form are admittedly somewhat naive. However, the theory serves as the basis for an initial study (Pereira & Romberg, in press), for the study reported here, and for a subsequent study now underway, and is thus in the process of evolving toward a higher level of sophistication.

Elements of the Potential Theory

Basic Ideas

These ideas are in the process of clarification, and new relationships among them are still being discovered. An attempt to formalize them into a theory will be appropriate only after the structure of their relationships has crystallized. Although this study could have been reported without them, hopefully their value as theoretical background will be seen to justify their inclusion.

Thinking is here defined as the subject's consciousness of the states of energy of the brain. Each state of energy makes

available registration of signs which come from the senses, the nervous system, the movements of the body, and the movements and functioning within the brain. The registration of these signs obeys special natural rules of codification and decodification. These rules will be referred to as universals which control the operations used by the brain for constructing messages in its sign system. Perceptions which generate brain signs are of three types: static, dynamic, and relational. Static perception is not thinking as it involves only a non-conscious "picturing" of an event. Dynamic perception involves constructing a mental representation of the main features of a sequence of events. Relational perception involves abstracting the relational properties from a sequence of events. Such abstract thinking can be done without mental representation.

Thinking's manifestation is dependent upon biological systems and has physical origin. The clearest manifestation of thinking is social language. Language in general is both a set of signs and the rules and operations defined for that set of signs in order to represent other signs of the same nature or of different nature for the purpose of communication. There are three types of language: natural, personal, and social. Language manifests itself primarily in the social interaction among human beings. It may be recognized through overt representations, which can be oral or written verbalization. Natural language is the language inside one's own brain. It

is the most powerful of all languages, since it is able to represent any of the changes of states of energy with great speed. Personal language is the language used by the subject himself for his intrapersonal communications. The personal language is dependent on the history of the subject and his social language. Social language is the language used in interpersonal communication. Intrapersonal communication has a four dimensional quality while social communication is linear. For this reason when intrapersonal communication is translated into social language, more time is needed for the whole communication. The biological conditions of the subject and the environment in which he lives suggest the set of signs which is most appropriate for his communications.

Social language may be either free or controlled, and controlled social language is of two kinds formal and quasi-formal. A social language of free construction is generated from the social interactions of the human beings in a specific environment. The signs and the rules are used with no pre-determination. States of group and individual consciousness generate the social language. This language is dependent on the history of the social group. The individual states of consciousness are assimilated slowly by the social language. The universals of the brain control the choices and conventions of the social group in an underlying way unknown to its members. Portuguese, Latin, French, and English are examples of social languages. Formal languages, such as those

constructed for computers, are built up by a social group with clear, conscious objectives and a priori rules and operations. The quasi-formal languages, e.g., the language of symbolic logic, are used for scientific communications.

Social language has at least the following characteristics:

a) Social language acts as a shifter from the planes of perception and action to the plane of mental representation or to abstract reasoning. The necessity for one to adapt himself to others creates between them a new order of reality (a new plane of representation). Here language holds sway, transforming into words those operations and relations which were previously the province of action. Language helps to make explicit the operations and relations which, although sufficient for the purpose of action, were only implicit.

b) Social language, a very economical way to transmit information, acts as an accelerator of the process of communication.

c) It acts as an analytical and synthetical tool in the acquisition of knowledge and the discovery of new ideas. Language aids in the realization of analysis and synthesis in the plane of mental representation since it brings to that level the characteristics of the biological level.

d) It acts as a catalyst in the thinking process.

e) It acts as a displacer in the communication process. One can represent something in language that is remote in space and time from the plane of speaking (or writing).

Social language among people is a necessary condition for the manifestation of natural reasoning processes. Overt verbalization is talking out, i.e., the set of signs is composed of sounds.

The dynamics of the construction of a language depend on three forces: analysis, synthesis, and motivation. Analysis is considered as a process which breaks down a set of signs into its constituent elements or parts so that the relationships between the signs are made explicit. It also determines a selection or classification of the signs. Three different types of analysis are possible: (1) natural analysis done by the organism without awareness of the subject (sensorial, motor, etc.), (2) mental analysis without awareness (use of the universals), and (3) representational analysis with awareness (operations in the mental representational plane). Synthesis is considered as a natural process which combines a set of signs making possible mental development. For each synthesis, the subject restructures precedent knowledge and acquires a force for further development. Synthesis is the creator of different levels of consciousness. Three types of synthesis are possible: (1) natural synthesis done by the organism without awareness of the subject (sensorial, motor, etc.), (2) mental synthesis without awareness (with the universals), and (3) representational synthesis with awareness (operations in the mental representational level). The motivation is a force resulting from imbalance of the biological or sensorial

system. This imbalance creates energetical conditions for initiation of the process of construction. Motivation, natural analysis, and natural synthesis determine all the processes of development of the subject and provide the elements for his evolution and learning.

Fundamental Propositions

(i) Intensity of thinking is directly proportional to a generative force and inversely proportional to resistance against its manifestation.

The generative force of thinking is determined by the psychological motivation (availability of the quantity of energy for manifestation) that a human being has for a certain set of signs. Resistance is related to the knowledge that the subject has of the set of signs and of its rules, the limitations of the defined language for expressing and representing the different states of consciousness, and the interference of the signs belonging to different systems and languages. The problem of resistance is fundamental in the learning process since energy cannot manifest itself freely in language if it is transformed into a force of inhibition which reduces the subject's powers of expression and motivation for learning.

(ii) The energy to be manifested in thinking is inversely proportional to the force of inhibition. Thinking that cannot be freely manifested in a specific language is transformed into

a force of inhibition, thus limiting the capability of expression and motivation of the subject.

Mathematical Background

The task to be learned in this study is a mathematical structure. The word "structure" has appeared frequently in mathematical literature and unfortunately means different things to different people. The idea of mathematical structure as well as the concepts of binary operation, group, abelian group, and Klein's Group used in the study are based on Nicholas Bourbaki's concept of mathematical structure (1968).

Given three distinct sets E, F, G, other sets may be formed from them by taking their sets of subsets, or by forming the product of one of them by itself, or again by forming the product of two of them in a certain order. In this way 12 sets are obtained. If these new sets are added to the three original sets E, F, G, the same operations may be repeated on these 15 sets (omitting those sets already obtained), and so on. In general, any one of the sets obtained by this procedure (according to an explicit scheme) is said to belong to the scale of sets on E, F, G, as base.

Consider a set M in a scale of sets whose base consists, for example, of three sets E, F, G. Also, assume that a certain number of explicitly stated properties of a generic element of M are given, and let T be the intersection of the subsets of M defined by

these properties. An element δ of T is said to define a structure of the species T on E, F, G. The structures of species T are, therefore, characterized by the scheme of formation of M from E, F, G and by the properties defining T, which are called the axioms of these structures. A specific name is given to all the structures of the same species. The same name is given to the structures which satisfy these axioms, independently of the set on which they are defined; and the propositions deduced from these axioms are valid in any set, because their formulation does not involve any special properties of the set E.

Usually when a scale is used with a base consisting of several sets E, F, G, one of these sets, say E, plays a preponderant role in the structures under consideration. Therefore, those structures are said to be defined in the set E, with F and G considered as auxiliary sets.

Finally, to simplify the language, a particular name is often given to a set which has been endowed with a structure of a definite species. Thus, the terms group, ring, field, lattice, etc., are used to denote sets endowed with certain structures.

When one is concerned with structures on a single set E, the bijection of E onto E' which transports δ and δ' is called an isomorphism of the set E, endowed with the structure δ, onto the set E', endowed with the structure δ'. An isomorphism of a set E, endowed with a structure δ, onto itself is called an automorphism.

Fraleigh provides the following definition: "A binary operation § on a set is a rule which assigns to each ordered pair of elements of the set some element of the set" (Fraleigh, 1969, p. 5). It should be observed that: (1) exactly one element is assigned to each possible ordered pair of elements of S and (2) for each ordered pair of elements of S, the element assigned to it is again in S.

Fraleigh gives the following definition of group:

> A group $\langle G, \S \rangle$ is a set G, together with a binary operation § on G, such that the following axioms are satisfied: Axiom 1. The binary operation § is associative. The operation § is associative if (and only if) (a § b) § c = a § (b § c) for all a, b, c ε G. Axiom 2. There is an element e in G such that e § x = x § e = e for all x ε G. This element e is an identity element for § on G. Axiom 3. For each a in G, there is an element a' in G with the property that a' § a = a § a' = e. The element a' is an inverse of a with respect to § (Fraleigh, 1969, p. 14).

A binary operation § on a set G is commutative if (and only if) a § b = b § a for all a, b ε G. A group G is abelian if its binary operation § is commutative.

For a finite set, a binary operation on the set can also be defined by means of a table. (ith entry on the left) § (jth entry on the top) = (entry in the ith row and jth column of the answers). For example, a table with the following properties defines an abelian group (Grossman, 1964):

1. Each element of the group must appear once and only once in each row and column of the table.

2. Within the table the intersection of the row containing the element x and the column containing the element y must be x § y. This property corresponds to Axiom 1.
3. One row within the table, namely, the row labeled by the symbol x, is identical with the row of symbols at the top of the table, and one column within the table, namely the column headed by the symbol x, is identical with the column of symbols at the left of the table. This property corresponds to Axiom 2.
4. Every symbol in the table can be associated with another symbol so that the row labeled by the first symbol, say x, and the column headed by its associate, call it y, intersect at entry z (the identity element); the row labeled y and the column headed by x also intersect at an entry y (the identity element), and these two entries y are symmetrically located with respect to the main diagonal. This property corresponds to Axiom 3.
5. The entries of the table are symmetric with respect to the diagonal which starts at the upper left corner of the table and terminates at the lower right corner. This property corresponds to the commutativity.

For the purpose of this study, Klein's Four Group will be defined by the following:

Table 2.1

OPERATIONAL TABLE FOR KLEIN'S FOUR GROUP

§	○	□	✡	△
○	○	□	✡	△
□	□	○	△	✡
✡	✡	△	○	□
△	△	✡	□	○

This table satisfies the five properties above; therefore the Klein's Four Group is an abelian group.

Psychological Background

The following explanations about terminal performance and retention, rules, strategies, and intelligence are important for a better understanding of this study.

Terminal Performance

From Table 2.1 there are 16 possible closed sentences of the form a o b = c. If any one of the elements (a, b, or c) is left blank, a set of 48 open sentences can be constructed. The terminal

test involved presenting to each subject in a random order this set of open sentences. The number of correct responses to this set of open sentences was used as a measure of terminal performance.

Retention

The following definition of retention, given by Deese (1958), was used in the study: retention refers to the extent to which material originally learned is retained. The level of retention was determined by measuring deviations from a fixed level of performance over a specified time interval (Deese, 1958, p. 236). The terminal test (T.T.) was reused as the retention test (R.T.) and administered three weeks after the terminal test.

Rules

The following three rules were considered for the problems.

$S = [\text{circle, square, star, triangle}]$

1. $\forall x \varepsilon S, \exists e \varepsilon S \mid x \circ e = e \circ x = x$ (Identity Rule)
2. $\forall x \varepsilon S, \exists x' \varepsilon S \mid x \circ x' = e$ (in the case $x = x'$) (Symmetric Rule)
3. $\forall x, y \varepsilon S \mid x \neq y \wedge x \neq e \wedge y \neq e \rightarrow \exists z \varepsilon S \mid z \neq e$
$\wedge z \neq x \wedge z \neq y \wedge x \circ y = z$ (Rule K)

The following conditions were considered as <u>necessary</u> conditions for a rule:

a. The subject plays <u>before</u> to get the rule using positive instances based on right or wrong predictions.

b. The subject plays after to confirm the rule using positive instances with only right predictions.

The subject was marked as having a rule if, in addition, one of the following conditions was observed:

a. The subject gives at least two consecutive plays using the same rule with right predictions.

b. The subject plays with instances of the same rule (at least three) in a discrete way with right predictions.

A subject probably has a weak rule if the following condition is observed: The subject does not play again with instances of the rule once he has established it. In this study it was assumed that human beings check the rules that they learn. Rules were analyzed by observing the sequence of predictions and by using graph analysis.

Strategies

The following definition of strategy given by Bruner (1956) was used: "A strategy refers to a pattern of decisions in the acquisition, retention, and utilization of information that serves to meet certain objectives, i.e., to insure certain forms of outcome and to insure against certain others" (Bruner, 1956, p. 54). In this study it was also assumed that a strategy is the result of a natural capability of the subject and is related to his mental operational development. Three types of strategies were considered: synthetical, analytical, and sensorial.

A synthetical strategy is characterized by the use of a rule in a continuous way when the subject has its mental representation. An analytical strategy is characterized by the discrete application of a rule when the subject does not have its defined mental representation. A sensorial strategy is characterized by both successes and failures in the use of a rule and the subject does not have any mental representation.

It was assumed that a subject could be synthetical in relation to certain rules, but analytical or sensorial in relation to other rules. It was further assumed that a synthetical strategy indicates that the subject has an operational development for the rule in use. Finally, it was assumed that in order to study strategies it is necessary to observe the behavior of the subject after he has discovered a rule by himself. Thus, the main problem was to identify when the subject acquired a rule and then to observe his behavior. The analysis of strategies was done by graph analysis.

Intelligence

The test of intelligence was administered as a consequence of Branca's affirmation that the performance of subjects on a structure scale was significantly related to intelligence (Branca, 1970, p. 96).

To get consistent intelligence scores, two scales from the Lorge-Thorndike Intelligence Test, Verbal Battery (Lorge & Thorndike, 1954) were used. These scales were used in the National Longitudinal

Study of Mathematical Abilities (Romberg & Wilson, 1969). The authors report a close correlation between verbal intelligence and mathematical ability:

> Verbal comprehension - this factor is nearly synonymous with verbal intelligence as measured by standard tests and includes verbal reasoning. Verbal scales in IQ batteries are usually used to measure this factor. In most cases, these verbal abilities have been found to relate to mathematics grades nearly as highly as the numerical abilities sections relate to the mathematics grades. These tests have been found to group together in factor analytic studies and are isolated separate elements in aptitude batteries (Romberg & Wilson, 1969, p. 164).

Then they elaborate on verbal reasoning:

> This dimension involves the ability to make inferences from verbally presented material and taps richness of vocabulary, verbal comprehension, and extensity of mediation constructs (Romberg & Wilson, 1969, p. 170).

Finally, they justify the use of the Lorge-Thorndike Intelligence Tests as follows:

> Scales from the Lorge-Thorndike Intelligence Tests were selected to tap the reasoning factor. The scales selected were Verbal Classification, Vocabulary, Verbal Analogy, Numerical Relationships and Pictorial Analogy. These scales were selected because existing evidence indicated that they would have the highest correlations with standard IQ scores (Romberg & Wilson, 1969, p. 170).

The Verbal Classification and the Verbal Analogy scales were used as a potential covariate in this study.

Related research studies are examined in the next chapter and classified in the following categories: (1) verbalization, (2) mathematical structure, and (3) strategy. Their main questions and findings are explained to make clear the differences between this study and earlier work.

Chapter III

RELATED RESEARCH

In this chapter past research on verbalization, structures, and strategies is presented. The search for relevant literature suggested the works of Hendrix, Retzer, Gagné, Palzere, Dienes, Branca, Bruner, and Hanfmann. Studies by these scholars are presented and analyzed from the perspective of the potential theory developed in Chapter II. Their findings support the notions that verbalization, structures, and strategies are important issues for research in mathematics education.

Hendrix's Study

The first experiment relating verbalization and the learning of mathematics was conducted by Gertrude Hendrix, who reported her findings in "A New Clue to Transfer of Training" (1947). She sought to determine the extent to which the way one learns a generalization affects the probability of his recognizing a chance to apply it. For her experiment she chose the rule, "The sum of the first n odd numbers is n-square," which was conveyed to each of three groups by a different method. The generalization was first stated to members of Group I and then verified by both teacher and subjects. Members of Group II were given several problems from which the rule could be generalized. The

rule was not verbalized, and each subject was asked to leave the room as soon as he manifested nonverbal evidence, such as a smile, which indicated that he had discovered it. Subjects in Group III followed the same procedure of self-discovery as those in Group II, but they were asked to verbalize the rule after recognizing it. Terminal testing supported the following three hypotheses:

> 1. For generation of transfer power, the unverbalized awareness method of learning a generalization is better t . method in which an authoritative statement of the generalization comes first.
>
> 2. Verbalizing a generalization immediately after discovery does not increase transfer power.
>
> 3. Verbalizing a generalization immedia ly after discovery may actually decrease transfer powe (Hendrix, 1947, p. 198).

Her interpretation of the data led to the following statement:

> . . . it is the intermediate flash of nonverbalized awareness that actually accounts for the transfer power . . . . Important as symbolic formulation must be for verification and organization of knowledge, it is not the key to transfer. The key is a sub-verbal internal process--something which must happen to the organism before it has any new knowledge to verbalize (Hendrix, 1947, p. 200).

She considered the proposition that discovery phenomena should be separated from the process of composing sentences which express those phenomena to be a significant breakthrough in pedagogical theory. And she advocates this separation again in "Learning by Discovery" (1961), treated in the Introduction.

Retzer's Study

Some mathematical educators contested Hendrix's conclusions. Retzer challenges her 1961 article, suggesting that instead of asking,

"How soon after discovering a generalization should a student verbalize?," she should have asked, "What is the student's facility with the language he needs to precisely verbalize his discovery?" Retzer (1969) performed an experiment designed to (1) test the effect of teaching certain concepts of logic on verbalization of discovered mathematical generalizations, (2) prepare a population which has demonstrated the ability to verbalize newly discovered mathematical generalizations with precision, and (3) test the effect of an ability to verbalize discovered mathematical generalizations upon the ability to use that generalization. His sample consisted of eighth-grade students enrolled in seven mathematics classes taught by three teachers. Students in Phase I completed a programed unit, Sentences of Logic. Phase II consisted of discovery programs using nonverbal awareness, followed by verbalization on the part of the student. The hypotheses tested in Phase I were:

> H_1: Completion of the Sentences of Logic unit has no effect on the ability of junior high school students to verbalize discovered mathematical generalizations.
>
> H_2: The ability level of junior high school students has no effect on their ability to verbalize discovered mathematical generalizations.
>
> H_3: The effect of the completion of the Sentences of Logic unit on verbalization ability is independent of the ability level of junior high school students (Retzer, 1969, p. 5).

The hypotheses tested in Phase II were:

> H_4: Verbalization of discovered mathematical generalizations has no effect on the ability of junior high school students to use the generalizations.

H_5: The ability to state discovered mathematical generalizations with precision has no effect on the ability of junior high school students to use the generalizations.

H_6: The effect of verbalizing discovered mathematical generalizations on ability of junior high school students to use the generalizations is independent of the ability to state the generalizations with precision (Retzer, 1969, p. 6).

None of these hypotheses were rejected in his conclusion.

Retzer considered the following two statements as the important outcomes of his research: (1) The ability to state precisely discovered mathematical generalizations can be manipulated for educational purposes, and (2) The teacher may make formation of linguistic ability an explicit part of the curriculum. Precise verbalization after the acquisition of some knowledge of logic was Retzer's objective and the above considerations yielded his main ideas.

Gagné and Smith's Study

The effects of verbalization during problem solving were explored by Gagné and Smith (1962). They were interested in the following two questions: (1) If we let the subject discover his own principles, in his own words, but require that he verbalize them, will this facilitate or interfere with problem solving? and (2) Is it possible to establish through differences in performance the effects of instructions to find and formulate verbally a general principle? They investigated these questions by measuring performance on a standard series of three-circle tasks of the sort employed by Ewert and Lambert (1932), transfer to a final six-disc task of this type, and the adequacy with which subjects could make verbal formulations of general principles. Specifically,

the experiment compared the performance of groups of subjects who solved two-, three-, four-, and five-disc problems successively, under four conditions representing combinations of two treatment variables: (a) a requirement to state verbally a reason for each move at the time it was made; and (b) instructions to search for a general principle which could be stated verbally after the tasks were performed.

Gagné and Smith used 28 boys in grades 9 and 10, who were assigned randomly to the following four experimental groups: (1) Group V-SS (Verbalizing, Solution Set) was instructed to state aloud why they were making each individual move at the time they made it. In addition, these subjects were instructed to try to think of a general rule by means of which they could tell someone how to solve these problems, which was to be solicited afterwards by the experimenter, (2) Group V (Verbalizing, No Solution Set) was required to verbalize a reason for each move, but were not instructed to try to formulate a general rule for solution, (3) Group SS (No Verbalizing, Solution Set) was not required to verbalize, but were instructed to try to formulate a rule, and (4) Group NO (No Verbalizing, No Solution Set) was simply told of the problem to be presented and its ground rules, with no additional instructions. Significant differences in number of moves were found between the scores of those groups which were required to verbalize and those which were not. Similar differences based on time scores were found in the contrast between verbalization and nonverbalization groups. Differences between other pairs of groups were not significant.

Gagné and Smith's results appear to indicate that requiring subjects to verbalize during practice has the effect of making them think of new reasons for their moves, and thus facilitates both the discovery of general principles and their employment in solving successive problems.

Palzere's Study

Palzere (1968) explored the effects of verbalization and nonverbalization after the student is aware of the concept related to the problem. The purpose of this study was to analyze the effect of verbalization and nonverbalization with respect to a student's ability to solve mathematical problems after having demonstrated that he is aware of the concept related to the problems. The following three hypotheses were formulated relating verbalization and nonverbalization.

H_1: There is no significant difference between the problem-solving ability of those students who do not verbalize a concept after demonstrating awareness of it and those who do verbalize this concept after demonstrating awareness of it.

H_2: There is no significant difference between the problem-solving ability of those students who do not verbalize a concept after demonstrating awareness of it and those who do verbalize a concept correctly after demonstrating awareness of it or with those who verbalize a concept incorrectly at first after demonstrating awareness of it and are brought to a correct verbalization.

H_3: There is no significant difference between the problem-solving ability of those students who do not verbalize a concept after demonstrating awareness of it and those students who verbalize a concept correctly after demonstrating awareness of it or those students who having verbalized a concept incorrectly after demonstrating awareness of it are forced to verbalize correctly or those students who having verbalized a concept incorrectly after demonstrating awareness of it are allowed to remain uncorrected (Palzere, 1967, p. 35).

Nine analyses of variance were carried out for the treatments. On the basis of the fact that only one of the nine tests was significant, he concluded:

> Even though the differences are not significant, there is a hint that verbalization should be encouraged after the student has demonstrated that he is aware of a concept (Palzere, 1967, p. 100).

This suggestion is in disagreement with Hendrix's conclusions.

Bruner, Goodnow, and Austin's Study

The works of Bruner, Dienes, Branca, and Hanfmann are pertinent to structures and strategies. Bruner, Goodnow, and Austin (1956) studied discernible strategies by which a person may proceed in his tasks. They labeled the four strategies as (1) the simultaneous-scanning strategy (the subject uses each instance encountered as an occasion for deducing which hypotheses are tenable and which have been eliminated); (2) the successive-scanning strategy (the subject tests a single hypothesis at a time); (3) the conservative-focusing strategy (the subject finds a positive instance to use as a focus, then makes a sequence of choices each of which alters but one attribute value of the first focus card and tests to see whether the change yields a positive or negative instance); and (4) the focus-gambling strategy (the subject uses a positive instance as a focus and then changes more than one attribute value at a time).

Dienes and Jeeves' Study

Dienes and Jeeves (1965) comment that Bruner concentrated principally upon description of overall strategy in the performance of the task given to the subjects and did not deal with the specific tactics used in the carrying out of the strategies. They conducted experiments with children and adults in order to answer the following questions: What individual strategies are distinguishable? and Do these naturally subdivide into types? They found the following three types of individual strategies: (1) the operational type (the subject appears to regard the element played as operating on the other element); (2) the pattern type (the subject appeared to regard the game as divided up into a certain number of sub-sections), and (3) the memory type (the subject stated that he has merely memorized all the different combinations). Their research was designed to answer the following six questions: (1) Under what conditions does transfer occur between structures?; (2) Under what circumstances are structures recognized as forming parts of other, more extensive structures?; (3) Under what circumstances will structures be generalized into more extensive structures, comprising the one already known?; (4) Is structure X learned and/or retained more easily if (a) X is learned with no reference to A or B?; (b) X is learned as a part of A?; (c) X is learned as a part of B?; (d) X is learned as a part of both A and B?; (5) With what kinds of properties must we endow a structure A, and not a structure B, so that, given the evidence for B, the structure A will

be expected?; and (6) Are the answers to any of the above questions different for children and adults, for males and females, or for children of different ages, or for adults of different ages? They did not arrive at definitive answers with their methodology.

Branca's Study

Branca (1970) explored their questions and pointed out new rules. His hypotheses were:

H_1: The distribution of evaluations for each of the group structure tasks will be in the same order as reported by Dienes and Jeeves (that is, in order of decreasing frequency or occurrence: Pattern, Memory, Operator).

H_2: The hierarchy of evaluations for each of the group structure tasks will be the same as the hierarchy reported by Dienes and Jeeves (that is, in order of decreasing efficiency: Operator, Pattern, Memory).

H_3: The subjects will be consistent in their evaluations and use of strategies across the group structure tasks. (Subjects who give a particular evaluation and use a particular strategy on one of the group tasks will tend to give the same evaluation and use the same strategy on the other group tasks.)

H_4: Subjects who give a particular evaluation and use a particular strategy on the tasks involving the group structure will tend to perform in a similar way on the network structure task.

Branca reported that all of his hypotheses were supported. He observed that intelligence was significantly related to performance. In conclusion Branca commented:

> Each of the tasks in the present study was restrictive in the sense that one element of the binary operation was determined for the subject and only one could be freely selected. . . . More exploratory work is necessary to determine why subjects make the moves they do and what information they regard as most important. . . . On the basis of further information, strategies

> might be identified that could be characterized by the moves a subject makes.
>
> The problems and questions raised by Dienes and Jeeves in Thinking in Structures have only begun to be explored. The present study has examined strategies in learning mathematical structures and has raised even more problems. Additional studies are encouraged to solve these problems and to raise still others (Branca, 1970, p. 104).

Hanfmann and Kasanin's Study

Hanfmann and Kasanin (1937) observed three strategies in the development of conceptual thinking: (1) categorical strategy--the subject identifies certain general characteristics, representative of certain categories; (2) insight into the multiple possibilities of the choice--the subject realizes that he does not know the basis of classification, that his task consists precisely in finding it by trying different possibilities; and (3) consideration of the total system--this strategy prompts the subject to test every general characteristic to see whether or not it will yield classes, and keeps him from establishing groups based on different principles and, therefore, not mutually exclusive. His actions are regulated by the nature of the task much more than by the rules of the experiment.

Pereira and Romberg's Study

In 1973, Pereira and Romberg (in press) carried out an experiment concerning the effects of verbalization on the learning of mathematical structures. Their objectives were as follows: (1) to explore the sensitivity of seven variables: prediction, construction of a table, facts, left-placeholder, right-placeholder, right answers in a terminal

test, and rules as dependent variables, (2) to explore the interaction between performance and sex, (3) to explore the possibilities of recognizing and categorizing the strategies used by the subjects in the acquisition of the rules, (4) to explore the possibilities for determining a set of rules which would point out when a subject acquired one of the rules of definition of the mathematical structure, (5) to explore the nature of the language used in the overt verbalization of the subjects, and (6) to explore the use of a new machine which was especially constructed for the experiment and had embodied in its circuit a mathematical structure.

In that study four instructional situations in which overt verbalization could be examined were considered: (1) subjects talk aloud while doing mathematical activities and then are silent after the activities, (2) subjects are silent while doing the activities and explain findings afterwards, (3) subjects verbalize both during and after they have completed activities, and (4) children do not verbalize either during or after they have completed mathematical activities. The population was composed of 24 children, 12 boys and 12 girls, of 11-12 years old. The experiment was carried out with each child individually. The subjects were randomly assigned to the four treatments with the same number of girls and boys for each treatment. Univariate and multivariate analyses of variance were carried out with the following results: (1) The most sensitive variable was right answers on the terminal test, (2) The interaction between sex and performance was significant, and (3) The boys had better performance in the treatments

with verbalization during learning activities and the girls had better performance with verbalization during questioning. Graph analysis was used to recognize the rules discovered from the subject's behavior.

The potential theory, the findings of Pereira and Romberg's study and the knowledge gleaned from past research motivated the investigator to plan this study.

Chapter IV

DESIGN AND PROCEDURE

In this chapter a detailed description of the development and execution of the study is given. The empirical propositions, the treatment, the machine used in the experiment, the tasks, the nature of the observations, the statistical hypotheses, and the population are explained.

Empirical Propositions

I. Subjects who learn in silence, without overt verbalization, during physical mathematical learning activities (actions upon physical objects), perform and retain better than subjects who overtly verbalize while learning.

This initial empirical proposition is a consequence of the two fundamental propositions of thinking.

II. If learning is done as established in Theorem I, then the process of discovering rules is accelerated. This second empirical proposition is a consequence of the first. It assumes that the processes of analysis and synthesis are fulfilled.

Treatments

In order to gather evidence for testing these theorems, an experiment was designed in which each subject was randomly assigned to one

of four instructional situations, or treatments. Subjects in all treatment groups performed the same sequence of five tasks, the last of which was a terminal test. Tasks I and III were devoted exclusively to learning; Tasks II and IV included specific questions about what and how the subject learned. By performing the first four tasks, the subject discovered the rules of the game, that is, the properties of Klein's Four Group. Treatments differed from each other according to the presence or absence of overt verbalization in the performance of tasks so that the effects of this variable on the learning process could be measured.

T_1: The subjects were required to *talk aloud* during Tasks I, II, III, and IV. (LQ)

T_2: The subjects were required to *work silently* (without talking) during Tasks I and III and to *talk aloud* during Tasks II and IV. $(\bar{L}Q)$

T_3: The subjects were required to *talk aloud* during Tasks I and III and to *work silently* (without talking) during Tasks II and IV. $(L\bar{Q})$

T_4: The subjects were required to *work silently* (without talking) during Tasks I, II, III, and IV. $(\bar{L}\bar{Q})$

Tasks

The five different tasks which were performed by each subject individually are described.

Task I. Subjects played for three minutes with a machine that has buttons with four figures (circle, square, star, and triangle). When they pushed a button with figures, one of the four figures was lighted on the right of the machine. The task was to discover how to light up each one of the four figures on the right of the machine

(see Figure 4.1). T_2 and T_4 subjects were instructed to work silently (without talking), but to think about all the reasons of their choices. T_1 and T_3 subjects were instructed to talk aloud about all the reasons for their choices or simply to describe what they did.

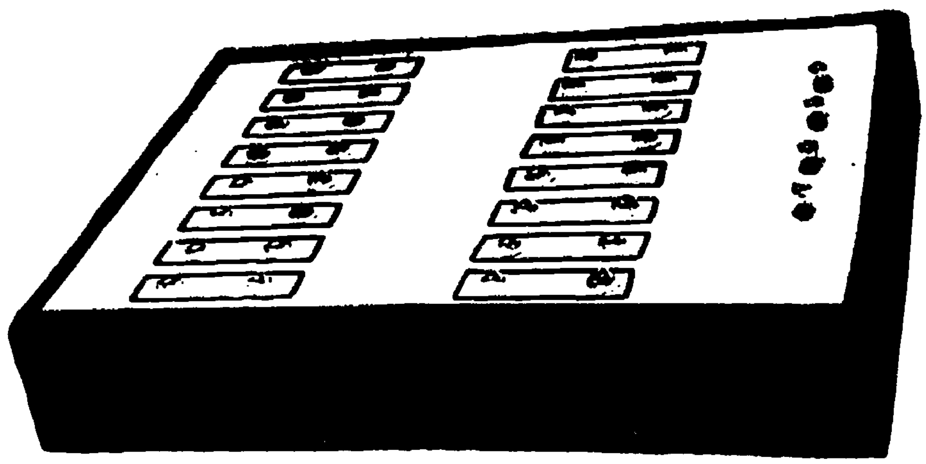

Figure 4.1. The display for the machine used in the study

Task II. The task was done in the following way:

1. The first of a series of questions was posed.
2. T_3 and T_4 subjects were instructed to work silently (without talking), but again they were asked to think about the answer to the question. T_1 and T_2 subjects were instructed to talk aloud about the answer to the question.
3. They were instructed to use the same strategy for the remaining questions.

The questions used were as follows:

1. What have you found out so far?
2. How did you discover the way to light up each one of

the four figures on the right of the machine?

3. How many buttons did you need to push for lighting up any one of the four figures?

4. When you push two buttons in the same box, do they always light the same figure?

Task III. Each subject played with the machine and was given sheets of paper for marking choices and predictions. Each sheet replicates the face of the machine (see Figure 4.2).

The subject's task was to discover the rules of the game. He was instructed to work in the following way:

1. Choose one pair of figures on the sheet of paper.
2. Mark the chosen pair of figures on the paper.
3. Predict the figure that the chosen pair will light on the machine.
4. Mark the predicted figure on the right of the paper.
5. Check the figure that was predicted by using the machine.
6. T_2 and T_4 subjects were instructed to work silently, but to think about all the reasons for their choices. T_1 and T_3 subjects were instructed to talk aloud about all the reasons for their choices or to describe what they did.
7. Subjects were then told to continue in the same way until they could predict correctly the figure that would be lighted by any pair of buttons with figures

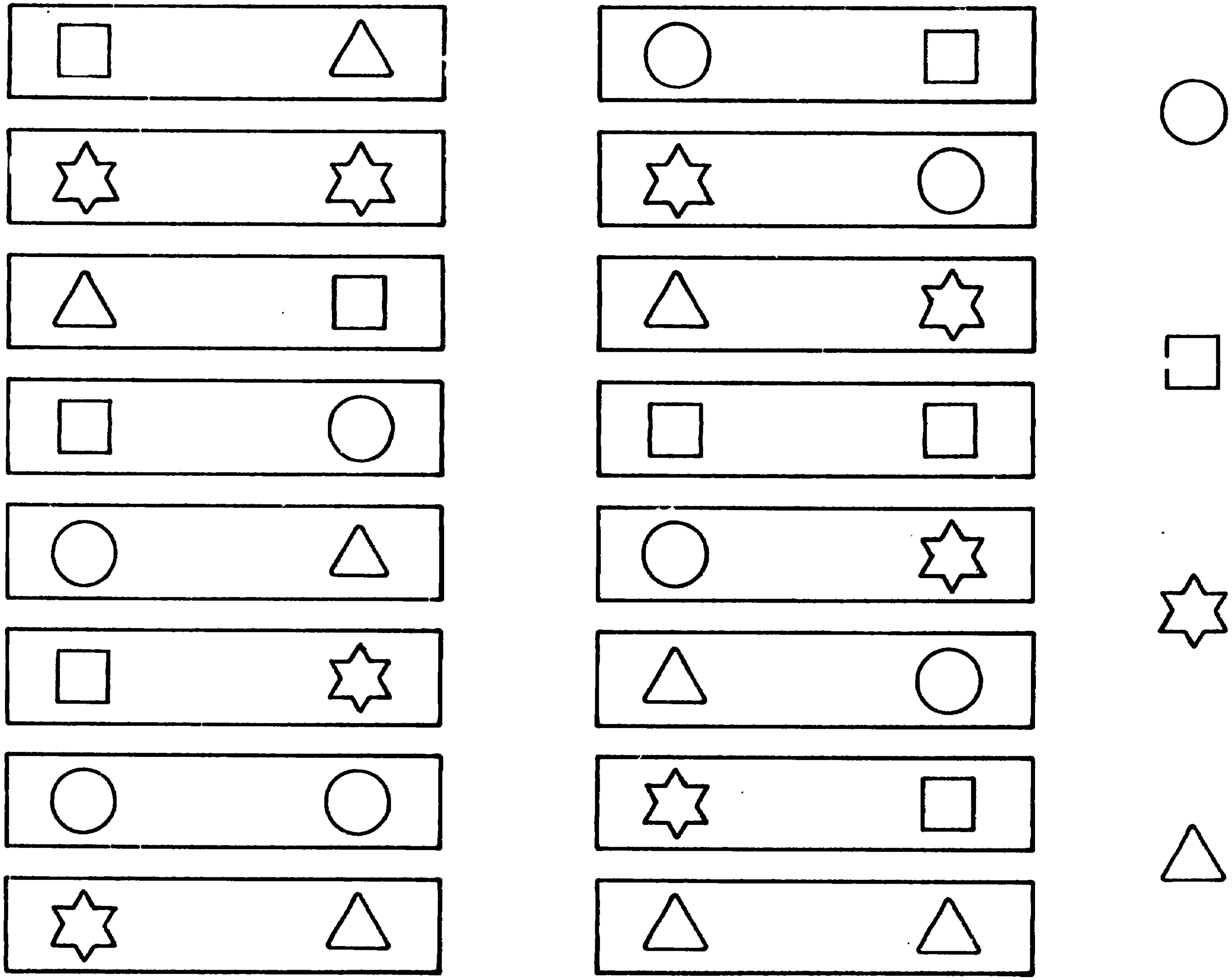

Figure 4.2. Display of choices on the machine.

on the machine. They were also instructed to keep in mind that <u>their main task was to get the rules of the game</u>.

8. Subjects were instructed to give all the sheets of paper to the experimenter.

<u>Task IV</u>. This task was done in the following way:

1. The first of a series of questions was posed.
2. T_3 and T_4 subjects were instructed to <u>work silently</u> (without talking), but were asked to think about the answer to the question. T_1 and T_2 subjects were instructed to <u>talk aloud</u> about the answer to the question.
3. They were instructed to do the same for the remaining questions.

The questions posed were:

1. What have you found out so far?
2. How did you get the correct predictions?
3. What pairs of figures made the circle light up?
4. What pairs of figures made the star light up?
5. What pairs of figures made the square light up?
6. What pairs of figures made the triangle light up?
7. If the first element of a pair was a circle and the answer was a square, what was the second element of the pair?
8. If the first element was a circle and the answer was a circle, what was the second element?

9. If the second element was a star and the answer was a triangle, what was the first element?
10. What did you learn about the circle in the game? About the square, star, and triangle?
11. What was the answer for the square and the star? What was the answer for the star and the square? What was the difference between the two pairs?
12. What were the rules of the game?
13. Did the rules of the game remind you of anything else you know?

Task V. This task was the terminal test (T.T.). Subjects were asked to predict the figure that would be lit by every pair printed in a list of exercises and to mark their predictions on the right of the sheet of paper.

The task was done in the following way:

1. Subjects were instructed to work silently (without talking).
2. They were asked to picture in their minds the figure not printed for a pair on the sheet of paper.
3. Then they were instructed to predict the figure not printed.
4. And finally, they were asked to mark the predicted figure on the right of the paper.
5. They were then told to do the same for all pairs.

Machine

The machine (see Figure 4.1) was built especially for the experiment. Its circuitry embodies Klein's Four Group. The stimuli are the following four figures of the same color: circle, square, star, and triangle. On the right of the machine four lights light up the following four figures: circle, square, star, and triangle.

Observations

The following diagram shows the order in which the observations were made.

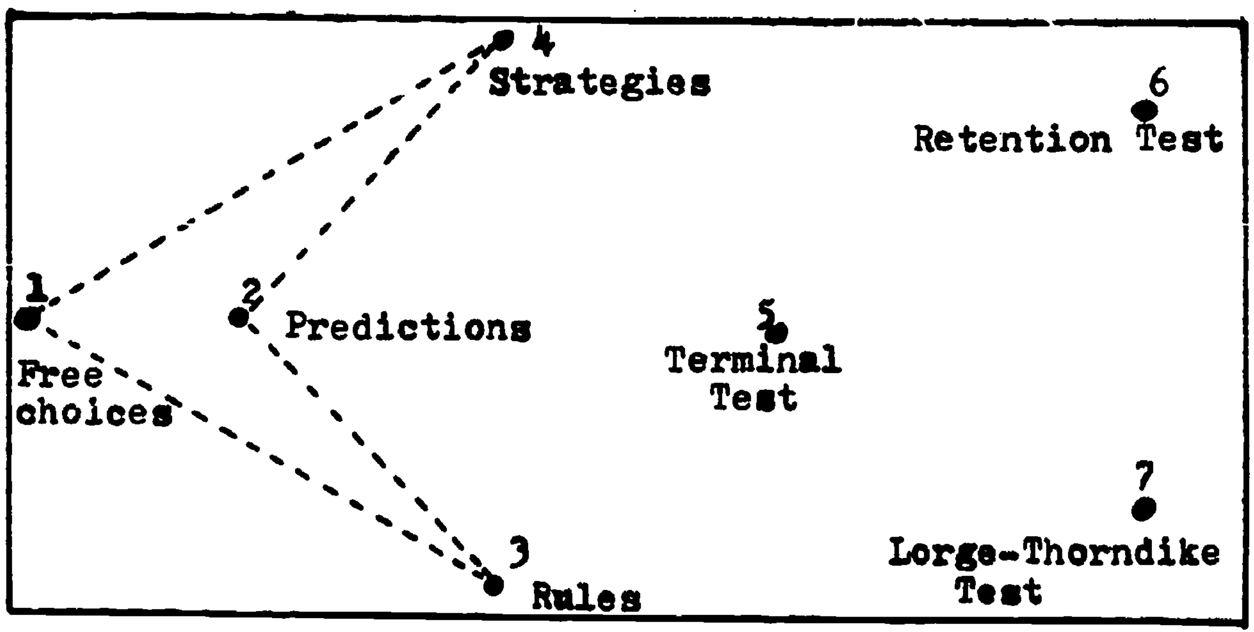

Figure 4.3. Research sequence

Choices

The subject's choices were observed while he was playing with the machine. He was free to choose any pair of figures from the 16 pairs displayed on the sheets of paper (see Figure 4.2). He was given 48 sheets of paper, and he marked one choice on each. He

could repeat the same choice as often as he wanted. The subject could discover the rules of the mathematical structure from his manipulations of the machine and observations during Tasks I and III.

Predictions

The subject made a prediction of the result for a chosen pair and marked his prediction on the sheet of paper before he used the machine to check his guess. He made 48 predictions, which were marked on the sheets of paper. Thus, the subject had 48 opportunities for learning the right answers.

Rules

The sequence of predictions and the sequence of choices were used in the analysis of the rules. The nature of the rules, the number of the rules, and the conditions needed to indicate from the subject's behavior that he had acquired a rule were presented in Chapter II.

Strategies

The subject's predictions and choices were used to recognize and categorize strategies. The definition, the classification and the method of analysis of the subjects' strategies are presented in Chapter II, and the respective analyses are presented in Chapter V.

Terminal Test (T.T.)

The terminal test (Appendix A) consisted of 48 items, which were composed of 16 facts, 16 symbol-open sentences with placeholder on the left and 16 symbol-open sentences with placeholder on the right. The

sequence of items was randomly arranged. The items corresponded to the following problems

$$a \circ b = ?, \quad ? \circ b = c, \quad \text{and } a \circ ? = c.$$

Each child took the test individually. The number of right answers on the test was used as an index of achievement.

Retention Test (R.T.)

The terminal test (T.T.) was reused as the retention test (R.T.). It was administered three weeks after the terminal test. Each child took the test. The definition of retention and the formula that was used for the scores of retention are explained in Chapter II.

Intelligence

The Lorge-Thorndike Intelligence Test, Verbal Battery, was used to ascertain the subjects' verbal aptitude. The explanation is presented in Chapter II and the computations in Chapter V. (See Appendix B for a copy of the test.)

Population

The subjects were 40 volunteer girls, 11-12 years old, from the Verona Elementary School, in Verona, Wisconsin. All had just completed the fifth grade. Each was randomly assigned to one of the four treatment groups. This procedure was used to assure that the treatment groups performed in a uniform experimental environment.

The study was carried out at the Reading Improvement Center, Verona, Wisconsin. It was done on two occasions.

Initial Time - Experiment and Terminal Test

Dates: July 23, 24, 25, 26, and 27

Times: 8:00 A.M. - 5:00 P.M.

Retention Time - Retention and Lorge-Thorndike Test

Dates: August 13, 14, 15, 16, 17, and 18

Times: 8:00 A.M. - 5:00 P.M.

Each girl provided her own transportation to the Reading Improvement Center. For participating in the experiment, each received $5.00 ($3.00 for the initial time and $2.00 for the retention time). All 40 girls selected participated in the experiment, but just 39 took the retention test. The one missing girl moved to Colorado and could not complete the experiment.

The facilities and the help of the staff of the Reading Improvement Center were excellent. Conditions for carrying out a highly controlled experiment could not have been better.

Statistical Hypotheses

To gather evidence for the theorems, the following statistical hypotheses were investigated:

H_1: The means of subjects' performance on a terminal test are the same whether they are required to talk aloud or work silently, without talking about their findings and to answer questions after mathematical-structural learning activities.

H_2: The means of subjects' performance on a terminal test

are the same whether they are required to talk aloud or work silently (without talking) during mathematical-structural learning activities.

H_3: There is no interaction effect between the means of subjects' performance on a terminal test whether they are required to talk aloud or work silently (without talking) during the same mathematical-structural activities.

H_4: The means of subjects' retention scores are the same whether they are required to talk aloud or work silently, without talking about their findings and to answer questions after mathematical-structural learning activities.

H_5: The means of subjects' retention scores are the same whether they are required to talk aloud or work silently (without talking) during mathematical-structural learning activities.

H_6: There is no interaction effect between the means of subjects' retention scores when they are required to talk aloud or work silently (without talking) during the same mathematical-structural activities.

H_7: The mean number of discovered rules is the same whether subjects are required to work silently (without talking) or talk aloud during mathematical-structural learning activities.

H_8: There is no relation between the nature of subjects'

discovered rules and their mean performance on the terminal test.

H_9: There is no relation between the mean <u>number</u> of subjects' discovered rules and their mean performance on a terminal test.

H_{10}: There is no relation between the nature of subjects' discovered rules and their mean performance on a retention test.

H_{11}: There is no relation between the mean number of subjects' discovered rules and their mean performance on the retention test.

The raw data are presented and the statistical analyses explained in Chapter V.

Chapter V

STATISTICAL ANALYSES

This chapter presents the statistical analyses used to test the hypotheses of this study, a comparison of two procedures for deriving rules, and the analyses of the subjects' strategies. The research hypotheses, initially stated in Chapter IV, are discussed within the section dealing with the appropriate dependent variable. The code and the input data used are presented respectively in Appendices C and D. Information is presented in the following sections: (1) Selection of a Covariate, (2) Terminal Test, (3) Retention Test, (4) Rules, and (5) Strategies.

Selection of a Covariate

Five related variables, IQ, Verbal A, Verbal B, Verbal Total, and Age, were used in a correlation analysis to select a covariate for the total score on the terminal test. Table 5.1 gives the descriptive statistics on these variables. Subject number 4 was excluded, because she did not take the second part of the experiment. The values of the correlation matrix are presented in Table 5.2. The possible covariates presented generally low correlations with the number of right answers on the terminal test. The verbal total

TABLE 5.1

DESCRIPTIVE STATISTICS FOR THE VERBAL IQ SCALES, AGE AND TERMINAL TEST

Variables	N Items	Mean (N = 39)	Standard Deviation
IQ	--	87.718	7.2872
Verbal A	25	10.564	4.4944
Verbal B	15	6.744	2.3810
Verbal Total (A+B)	40	17.308	6.0313
Age (CA-months)	--	137.490	4.6389
Terminal Test	48	23.615	12.0060

TABLE 5.2

CORRELATIONS BETWEEN THE TERMINAL TEST AND THE VERBAL-IQ SCALE

Variable	IQ	Verbal A	Verbal B	Verbal Total	Age (CA)
Terminal Test	.403	.406	.336	.436	.001

was chosen as covariate because it had the highest correlation with terminal test scores. A test of association was considered important for a final decision; Table 5.3 presents the results.

TABLE 5.3

TEST OF ASSOCIATION BETWEEN THE VERBAL TOTAL SCORE AND THE TERMINAL TEST

Variable	Square R	R	F	p <
TT	.2676	.5173	12.4201	.0013

The degree of freedom for the hypothesis tested was 1; the degree of freedom for error was 34. The F value for testing the hypothesis of no association between dependent and independent variable was 21.8096 and the degrees of freedom 2 and 33 with p less than .05. The hypothesis of no association is rejected. Thus, Verbal Total score was used as a covariate.

Terminal Test

The terminal test is presented in Appendix A and the raw data in Appendix C. Table 5.4 presents the identification of the cells.

TABLE 5.4

CELL IDENTIFICATION FOR THE FOUR TREATMENTS USED IN THIS STUDY

		LEARNING	
		Talk	No talk
QUESTIONING	Talk	LQ T_1	$\bar{L}Q$ T_2
	No talk	$L\bar{Q}$ T_3	$\bar{L}\bar{Q}$ T_4

The descriptive statistics for the terminal test are reported in Table 5.5. They are the observed cell means and cell standard deviations.

TABLE 5.5

STATISTICS DESCRIBING THE PERFORMANCE OF THE FOUR TREATMENT GROUPS ON THE TERMINAL TEST (T.T.)

Cell	Number of Items	Number of Subjects	Mean	Standard Deviation
LQ	48	10	17.300	3.713
$L\bar{Q}$	48	10	21.600	10.946
$\bar{L}Q$	48	10	31.700	14.080
$\bar{L}\bar{Q}$	48	10	23.889	13.214

Since a student could have gotten 12 items correct by guessing, the amount of learning (A_{Li}) is defined by the following formula for this terminal test.

$$A_{Li} = \frac{\bar{X}_i - 12}{36} \times 100$$

The following table presents the amount of learning for each one of the four treatment groups. Although the study was not interested in mastery of learning, the differences in the amount learned by the groups are of interest.

TABLE 5.6

AMOUNT OF LEARNING FOR EACH TREATMENT GROUP

Treatment Group	A_{L_i}
T_1 (LQ)	14.72
T_2 ($\bar{L}$Q)	54.72
T_3 (L$\bar{Q}$)	26.67
T_4 ($\bar{L}\bar{Q}$)	33.01

The following null hypotheses were tested using the adjusted cell means:

H_1: The means of subjects' performance on a terminal test are the same whether they are required to talk aloud or work silently, without talking about their findings and to answer questions after mathematical-structural learning activities.

H_2: The means of subjects' performance on a terminal test are the same whether they are required to talk aloud or work silently (without talking) during mathematical-structural learning activities.

H_3: There is no interaction effect between the means of subjects' performance on a terminal test whether they are required to talk aloud or work silently (without talking) during the same mathematical-structural activities.

Table 5.7 presents the ANCOVA table with the statistics to test each of the above three hypotheses.

TABLE 5.7

ANCOVA TABLE FOR TREATMENT FACTORS ON THE TERMINAL TEST

Source	df	MS	F	p <	Hypotheses
Talk during Learning	1	560.0848	5.9312	.0203	H_2
Talk during Questioning	1	47.3455	.5014	.4833	H_1
Interaction	1	596.3714	6.3154	.0169	H_3
Residual	34	94.4309			

For H_1 the F value (with 1 and 34 degrees of freedom) is .5014 with p less than .4833. Hypothesis H_1 cannot be rejected at $\alpha = .05$. For H_2 the F value (with 1 and 34 degrees of freedom is 5.9312 with p less than .0203. Hypothesis H_2 is rejected at $\alpha = .05$. And, for H_3 the F value (with 1 and 34 degrees of freedom) is 6.3154 with p less than .0169. Hypothesis H_3 is rejected at $\alpha = .05$.

Retention Test

The second variable was the number of right answers on the retention test. Table 5.8 presents the descriptive statistics for the retention test.

TABLE 5.8

STATISTICS DESCRIBING THE PERFORMANCE OF THE FOUR TREATMENT GROUPS ON THE RETENTION TEST (R.T.)

Cell	Number of Items	Number of Subjects	Mean	Standard Deviations
T_1 (LQ)	48	10	14.9000	2.8461
T_2 ($\bar{L}$Q)	48	10	35.2000	13.3816
T_3 (L$\bar{Q}$)	48	10	21.7000	12.005
T_4 ($\bar{L}\bar{Q}$)	48	9	22.1111	9.2797

Table 5.9 presents the statistics to test the hypothesis of no association between the retention test and the covariate (Verbal Total).

TABLE 5.9

TEST OF ASSOCIATION BETWEEN VERBAL TOTAL SCORE AND THE RETENTION TEST

Variable	Square R	R	F	P <
RT	.5631	.7504	43.8237	.0001

The F value (with 1 and 34 degrees of freedom) is 43.8237 with p less than .0001. The hypothesis of no association between the retention test and the covariate is rejected at $\alpha = .05$.

The following three null hypotheses were tested:

H_4: The means of subjects' retention scores are the same whether they are required to talk aloud or work silently, without talking about their findings and to answer questions after mathematical-structural activities.

H_5: The means of subjects' retention scores are the same whether they are required to talk aloud or work silently (without talking) during mathematical-structural learning activities.

H_6: There is no interaction effect between the means of subjects' retention scores whether they are required to talk aloud or work silently (without talking) during the same mathematical-structural activities.

Table 5.10 presents the ANCOVA table with the statistics to test each of the above three hypotheses.

TABLE 5.10

ANCOVA TABLE FOR TREATMENT FACTORS ON THE RETENTION TEST

Source	df	MS	F	p <	Hypotheses
Talk during Learning	1	860.4206	18.2481	.0002	H_5
Talk during Questioning	1	132.4646	2.8093	.1029	H_4
Interaction	1	1465.0437	31.0711	.0001	H_6
Residual	34	47.1513			

For H_4 the F value (with 1 and 34 degrees of freedom) is 2.8093 with p less than .1029. Hypothesis H_4 cannot be rejected at $\alpha = .05$. For H_5 the F value (with 1 and 34 degrees of freedom) is 18.2481 with p less than .0002. Hypothesis H_5 is rejected at $\alpha = .05$. And for H_6 the F value (with 1 and 34 degrees of freedom) is 31.0711 with p less than .0001. Hypothesis H_6 is rejected at $\alpha = .05$.

Rules

The third dependent variable was the number of rules discovered by the subjects. Appendix E presents the output data done by the computer with the use of the two methods of deriving rules. The consecutiveness method was described in Chapter IV. The ratio method consists of the determination of the ratio between the number of right and wrong predictions within a given rule. If the ratio is greater than 1, then the subject is considered to have discovered the rule. The consecutiveness method was used to test the statistical hypotheses and to analyze the subjects' strategies, because it provides more information about how the subjects got the rules. The Pearson Product-Moment Correlation Coefficient between the two methods was .7975.

Table 5.11 presents the descriptive statistics for the number of rules derived as ascertained by the computer with the consecutiveness method. Table 5.12 presents the statistics to test the hypothesis of no association between the number of rules and the covariate.

Table 5.11

STATISTICS DESCRIBING THE NUMBER OF RULES DISCOVERED

BY THE FOUR TREATMENT GROUPS

Cell	Number of Subjects	Mean	Standard Deviation
T_1 (LQ)	10	.4000	.5164
T_2 ($\bar{L}Q$)	10	1.2000	1.1353
T_3 ($L\bar{Q}$)	10	.6000	.8453
T_4 ($\bar{L}\bar{Q}$)	10	1.0000	1.0000

TABLE 5.12

TEST OF ASSOCIATION BETWEEN VERBAL TEST SCORE

AND THE NUMBER OF RULES DISCOVERED

Variable	Square R	R	F	p <
N Rules	.0150	.1225	.5176	.4768

The degrees of freedom for the hypothesis and error are respectively 1 and 34. The F value (with 1 and 34 degrees of freedom) is .5176 with p less than .4768. The hypothesis of no association between the number of rules and the covariate variable cannot be rejected. Thus, no covariate was used to test the hypothesis about rules.

Table 5.13 presents the statistics to test the following hypothesis:

H_7: The mean number of discovered rules is the same whether subjects are required to <u>work silently</u> (without talking) or <u>talk aloud during</u> mathematical-structural learning activities.

TABLE 5.13

ANCOVA TABLE FOR LEARNING FACTORS ON THE NUMBER OF RULES DISCOVERED

Source	df	MS	F	p <	Hypothesis
Talk during Learning	1	3.6000	4.5634	.0396	H_7
Error	36	.7889			

For H_7 the F value (with 1 and 34 degrees of freedom) is 4.5634 with p less than .0396. Hypothesis H_7 is rejected at $\alpha = .05$.

Table 5.14 presents the statistics for testing the following two hypotheses:

H_8: There is no relation between the <u>nature</u> of subjects' discovered rules and their mean performance on a terminal test.

H_{10}: There is no relation between the <u>nature</u> of subjects' discovered rules and their mean performance on a retention test.

TABLE 5.14

MULTIVARIATE REGRESSION ANALYSIS:

RELATION OF THE SPECIFIC RULES DISCOVERED TO THE TERMINAL TEST AND TO THE RETENTION TEST

Variables	Regr. Coef.	P. Corr. Coef.	Partial $F_{1,32}$	p Value	Hypotheses
Identity	5.086	.201	1.3543	.2531 H_A	H_8 (TT)
Symmetric	9.706	.457	8.4354	.0066 H_B	
Rule K	7.786	.345	4.3294	.0455 H_C	
Identity	-1.419	-.056	.0996	.7543 H_M	H_{10} (RT)
Symmetric	6.383	.312	3.4468	.0726 H_N	
Rule K	8.073	.348	4.3973	.0440 H_P	

Hypothesis H_8 is rejected since H_A cannot be rejected at $\alpha = .05$. The nature of the rule has influence in relation to terminal test performance. Hypothesis H_{10} of no relation of the nature of the rule to the retention test is also rejected, since H_M and H_N cannot be rejected at $\alpha = .05$, but H_P is rejected at $\alpha = .05$. The nature of the rule has influence in relation to the retention test performance.

Table 5.15 presents the statistics for testing the following two hypotheses:

H_9: There is no relation between the mean <u>number</u> of subjects' discovered rules and their mean performance on a terminal test.

H_{11}: There is no relation between the mean number of subjects' discovered rules and their mean performance on a retention test.

TABLE 5.15

MULTIVARIATE REGRESSION ANALYSIS:

RELATION OF THE TOTAL NUMBER OF RULES DISCOVERED TO THE TERMINAL TEST AND TO THE RETENTION TEST

Variables	Regr. Coef.	P. Corr. Coef.	Partial $F_{1,34}$	p Value	Hypotheses
NRules	8.021	.646	24.3030	.0000	H_9
NRules	5.275	.464	9.3302	.0044	H_{11}

For H_9 the F value (with 1 and 34 degrees of freedom) is 24.3030 with p value equal to .0000. The H_9 of no relation of the number of rules to the terminal test is rejected at $\alpha = .05$. And for H_{11} the F value (with 1 and 34 degrees of freedom) is 9.3302 with p value equal to .0044. Hypothesis H_{11} of no relation of the number of rules to the retention test is rejected at $\alpha = .05$.

The following six null hypotheses were included in this study in order to analyze the individual relations between identity, symmetric and Rule K to the terminal test and to the retention test.

H_{12}: There is no relation between the identity score and the mean on a terminal test.

H_{13}: There is no relation between the identity score and the mean on a retention test.

H_{14}: There is no relation between the symmetric score and the mean on a terminal test.

H_{15}: There is no relation between the symmetric score and the mean on a retention test.

H_{16}: There is no relation between the Rule K score and the mean on a terminal test.

H_{17}: There is no relation between the Rule K score and the mean on a retention test.

Table 5.16 presents the statistics for testing the above hypotheses.

TABLE 5.16

MULTIVARIATE REGRESSION ANALYSIS:

RELATION OF THE SPECIFIC RULES DISCOVERED TO THE TERMINAL TEST AND TO THE RETENTION TEST

Variable	Regr. Coef.	P. Corr. Coef.	Partial F 1,34	p Value	Hypotheses
Identity (TT)	9.073	.284	2.9780	.0935	H_{12}
Identity (RT)	1.723	.059	.1184	.7329	H_{13}
Symmetric (TT)	12.959	.565	15.9129	.0003	H_{14}
Symmetric (RT)	8.826	.420	7.2966	.0107	H_{15}
Rule K (TT)	12.426	.479	10.1408	.0031	H_{16}
Rule K (RT)	10.572	.446	8.4296	.0064	H_{17}

For H_{12} the F value (with 1 and 34 degrees of freedom) is 2.9780 with p less than .0935. Hypothesis H_{12} of no relation of identity to the terminal test cannot be rejected at $\alpha = .05$. For H_{13} the F value (with 1 and 34 degrees of freedom) is .1184 with p less than .7329. Hypothesis H_{13} of no relation of identity to the retention test cannot be rejected at $\alpha = .05$. For H_{14} the F value (with 1 and 34 degrees of freedom) is 15.9129 with p less than .0003. Hypothesis H_{14} of no relation between symmetric and the terminal test is rejected at $\alpha = .05$. For H_{15} the F value (with 1 and 34 degrees of freedom) is 7.2966 with p less than .0107. Hypothesis H_{15} of no relation between symmetric and the retention test is rejected at $\alpha = .05$. For H_{16} the F value (with 1 and 34 degrees of freedom) is 10.1408 with p less than .0031. Hypothesis H_{16} of no relation of the Rule K to the terminal test is rejected at $\alpha = .05$. And for H_{17} the F value (with 1 and 34 degrees of freedom) is 8.4296 with p less than .0064. Hypothesis H_{17} of no relation of the Rule K to the retention test is rejected at α .05.

Strategies

The definitions presented in Chapter IV were applied to identify and categorize the subjects' strategies. Appendix E lists the rules discovered by the subjects as computed with the consecutiveness method at the Madison Academic Computing Center. Table 5.17 presents the strategies employed by all subjects who discovered rules. The analytical and synthetical strategies were

TABLE 5.17

IDENTIFICATION OF STRATEGY USED TO DISCOVER RULES FOR ALL SUBJECTS WHO DISCOVERED RULES

ID.	Identity	Symmetric	Rule K
01	0	s	0
02	0	a	0
03	0	s	0
04	s	0	0
11	s	s	s
13	s	0	0
14	s	0	0
16	0	s	s
19	0	0	s
21	0	s	0
25	0	s	0
26	0	s	a
27	0	0	a
31	s	a	s
32	0	s	s
34	0	a	s
35	s	a	0
37	0	s	0
38	0	s	s
39	0	0	a
40	0	a	a

coded a and s, respectively. The operation § (Table 2.1) was used to mark the predictions of the subjects for Task III. The numbers which appear in Figure 5.1 indicate the order of the subject's choices for Task III. The number without a bar indicates a right prediction; the number with a bar indicates a wrong prediction. For example, $\bar{1}$ in Figure 5.1 indicates that the subject incorrectly chose as his first prediction the pair (star, star). The figures in the vertical column of the table are the first element of the pair; the figures in the horizontal column are the second element of the pair. The number 22 in Figure 5.1 indicates that the subject's 22nd choice was the pair (star, square), and that his prediction was correct. The heavy line in Figure 5.1 joining point 39 to 40 indicates that the subject made two consecutive right predictions for the same rule (symmetric, in this case). The broken line from point 40 to point 44 indicates that the subject gave right predictions for the same rule, but not consecutively. Figure 5.1 also shows that the subject discovered the symmetric rule, using the synthetical strategy. The graph analyses used to determine the strategies for each subject are presented in Appendix J.

Table 5.18 presents a summary of the number of rules discovered using each strategy.

The summary, the discussion of the findings, the limitations of the study, the conclusions related to the theorems, the implications for the naive theory, and the directions for new research are presented in Chapter VI.

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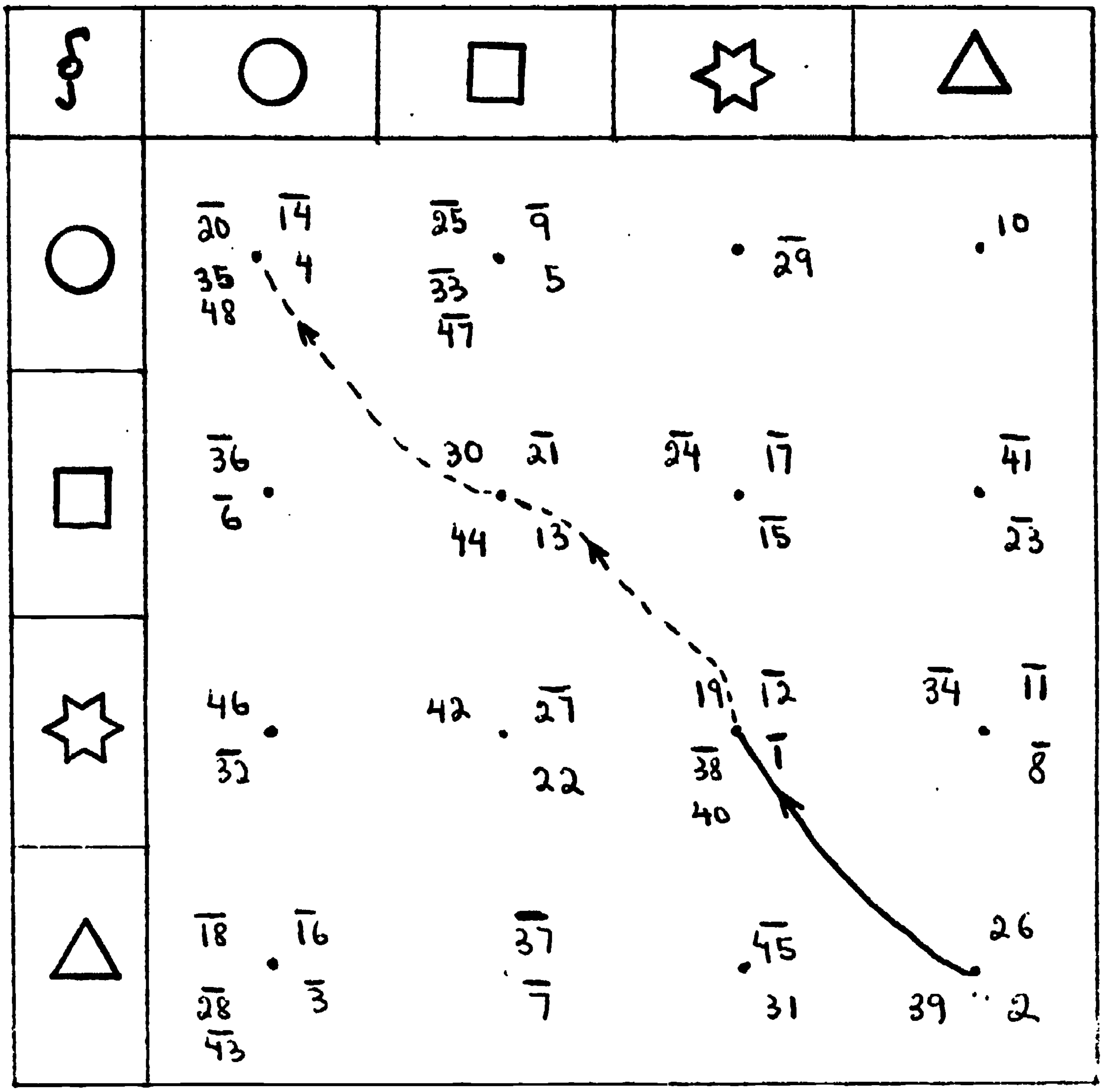

Figure 5.1. Graphical display of the predictions of Subject III during Task III

TABLE 5.18

SUMMARY OF RULES DISCOVERED

STRATEGIES	Identity	Symmetric	Rule K	Total
Analytical	0	5	4	9
Synthetical	6	10	7	23
Total	6	15	11	32

Chapter VI

DISCUSSION AND IMPLICATIONS

In this chapter a brief summary of the study is presented. Then the following eleven topics are discussed: (1) Limitations of the Study, (2) Learning During Task III, (3) Test Information, (4) Conclusions Related to Empirical Proposition I, (5) Conclusions Related to Empirical Proposition II, (6) Conclusions Related to the Rules, (7) Comparisons Among the Treatments, (8) Conclusions Related to the Subjects' Strategies, (9) Implications for Theory, (10) Implications for Curriculum Development, and (11) Recommendations for New Research.

Summary

The objective of the study was fourfold: (1) to describe the basis of a potential mathematics learning theory founded on the relationship between language and learning, (2) to gather evidence to test the validity of a theorem relating subjects' overt verbalization and performance after they have been taught a mathematical structure, (3) to gather evidence to test the validity of a theorem relating the subjects' overt verbalization and the number of discovered rules which define the mathematical structure, and (4) to examine the analytic strategies the subjects used in the learning of these rules.

The naive theory presented in Chapter II can be divided into the following four categories: (a) Basic Notions, (b) Fundamental Propositions, (c) Forces, and (d) Empirical Propositions.

(a) Basic Notions

a.1 Thinking is a special sequence of changes in the states of energy in the brain.

a.2 Language is a tool used for the expression of thought.

a.3 Social language has at least the five following characteristics:

a.3.1 It acts as a shifter from the planes of perception and action to the plane of mental representation.

a.3.2 It acts as an accelerator of the process of social communication.

a.3.3 It acts as an analytical and synthetical tool in the acquisition of knowledge and discovery of new ideas.

a.3.4 It acts as a catalyst in the thinking process.

a.3.5 It acts as a displacer in the communication process.

a.4 Overt verbalization among people is a necessary condition for the manifestation of natural reasoning processes.

(b) Fundamental Propositions

b.1 The intensity of thinking is directly proportional to a generative force and inversely proportional to resistance against its manifestation.

b.2 The generative force of thinking is determined by the psychological motivation that a human being has for a certain set of signs.

b.3 Resistance is considered to be the product of the knowledge that the subject has of the set of signs and of its rules, the limitations of the defined language for expressing and representing the different states of consciousness, and the interference of the signs belonging to different systems and languages.

b.4 The energy to be manifested for thinking is inversely proportional to a force of inhibition.

(c) Forces

Analysis, synthesis, and motivation are the three basic forces. Motivation gives permanence to a state; analysis gives the elements for a new synthesis which works as a creative force of the new levels.

(d) Empirical Propositions

d.1 If subjects learn in silence, without overt verbalization, during physical mathematical learning activities (actions upon physical objects), then the subjects perform better and retain more than subjects who overtly verbalize while learning.

d.2 If learning is done as established in Theorem I, then the process of discovering rules is accelerated.

Forty girls aged 11-12 years and residing in Verona were randomly assigned to the project and individually asked to participate in one of the four instructional situations: (a) Subjects verbalize both during and after they have done the activities (LQ), (b) Subjects are silent while doing the activities and afterwards answer questions and explain findings ($\bar{L}Q$), (c) Subjects talk aloud while doing mathematical activities and are silent after the activities ($L\bar{Q}$), and (d) Subjects do not verbalize either during or after they have done the activities ($\bar{L}\bar{Q}$).

The method used consisted of presenting the subject with pairs of geometric figures on buttons on a machine. The figures were replicated in a vertical column on the right of the machine and could be lit by pressing combinations of buttons. The machine was wired to embody a group structure, in this case, Klein's Four Group. The actual stimuli were pairs of four geometric figures: a circle, a square, a star, and a triangle. By manipulating the machine, the subject is led to discover the following three rules:

S = [circle, square, star, triangle]

(1) $\forall x \in S, \exists e \in S \mid x \circ e = e \circ x = x$ (Identity Rule)

(2) $\forall x \in S, \exists x^1 \in S \mid x \circ x^1 = e$(in the case $x = x^1$) (Symmetric Rule)

(3) $\forall x, y \in S \mid x \neq y \wedge x \neq e \wedge y \neq e \rightarrow \exists z \in S \mid z \neq e$

$\wedge z \neq x \wedge z \neq y \wedge x \circ y = z$ (Rule K)

In the study it was assumed that a strategy is the result of a natural capability of the subject and is related to his mental operational development. Three types of strategies were considered: synthetical, analytical, and sensorial.

A synthetical strategy is characterized by the use of a rule in a continuous way when the subject has its mental representation. An analytical strategy is characterized by the use of a rule in a discrete way when the subject does not have its defined mental representation. A sensorial strategy is characterized by success and failure in the use of a rule when the subject does not have any mental representation.

In Chapter V the following 17 hypotheses were analyzed for getting evidence to test the validity of the two theorems presented above.

H_1: The means of subjects' performance on a terminal test are the same whether they are required to talk aloud or work silently, without talking about their findings, and to answer questions after mathematical-structural learning activities.

H_2: The means of subjects' performance on a terminal test are the same whether they are required to talk aloud or work silently (without talking) during mathematical-structural learning activities.

H_3: There is no interaction effect between the means of subjects' performance on a terminal test whether they are required to talk aloud or work silently (without talking) during the same mathematical-structural activities.

H_4: The means of subjects' retention scores are the same whether they are required to talk aloud or work silently, without talking about their findings, and to answer questions after mathematical-structural activities.

H_5: The means of subjects' retention scores are the same whether they are required to talk aloud or work silently (without talking) during mathematical-structural learning activities.

H_6: There is no interaction effect between the means of subjects' retention scores whether they are required to talk aloud or work silently (without talking) during the same mathematical-structural activities.

H_7: The mean number of discovered rules is the same whether subjects are required to work silently (without talking) or talk aloud during mathematical-structural learning activities.

H_8: There is no relation between the nature of subjects' discovered rules and their mean performance on a terminal test.

H_9: There is no relation between the mean number of subjects' discovered rules and their mean performance on a terminal test.

H_{10}: There is no relation between the nature of subjects' discovered rules and their mean performance on a retention test.

H_{11}: There is no relation between the mean number of subjects' discovered rules and their mean performance on a retention test.

H_{12}: There is no relation between the identity score and the mean on a terminal test.

H_{13}: There is no relation between the identity score and the mean on a retention test.

H_{14}: There is no relation between the symmetric score and the mean on a terminal test.

H_{15}: There is no relation between the symmetric score and the mean on a retention test.

H_{16}: There is no relation between the Rule K score and the mean on a terminal test.

H_{17}: There is no relation between the Rule K score and the mean on a retention test.

The following section discusses the sources of internal and external validity of the study.

Limitations

While the sources of internal validity were controlled through the design of the study, some of the sources of external validity which permit generalization could not be controlled. The first source of external validity which was not controlled was the interaction of selection and treatment. Because of the great difficulty in obtaining subjects for the study during vacation time, the subjects were volunteers from the Elementary School of Verona, Wisconsin.

Thus, the results can only be interpreted for this population. A second source of invalidity may have been what Campbell and Stanley (1963) call reactive arrangements: "the patent artificiality of the experimental setting and the student's knowledge that he is participating in an experiment" (p. 20). An effort was made to have a normal setting, but the subjects worked in a special quiet room with great comfort and air-conditioned in the Reading Center of Verona. Thus, the reactive effects of the experimental arrangements preclude generalization about the effect of the experimental variable upon persons being exposed to it in non-experimental settings.

Learning During Task III

During Task III subjects were predicting which light would be lit when two buttons were pushed. Table 6.1 presents the total number of correct predictions for each treatment group out of a total of 480 predictions (10 subjects x 48 predictions).

TABLE 6.1

TOTAL NUMBER OF CORRECT PREDICTIONS DURING TASK III FOR EACH TREATMENT GROUP

Treatment	Number of Correct Predictions
T_1 (LQ)	161
T_2 ($\bar{L}Q$)	233
T_3 ($L\bar{Q}$)	160
T_4 ($\bar{L}\bar{Q}$)	222

Although no test for statistically significant differences was carried out, the total number of correct predictions for those who did not talk during learning (455) is considerably higher than for those who did (321).

Test Information

In interpreting the results, one must also consider the reliability of the instruments of measurement. The reliabilities of the terminal test and the retention test were respectable, as can be seen from the analyses that follow using Hoyt's test of reliability estimated by analysis of variance (1941).

The terminal test (Appendix A) consisted of 48 items. The complete item data are in Appendix F. The Hoyt's reliability coefficient of the terminal test was equal to .935; the standard error of measurement was 2.97.

The terminal, reused as the retention test, was readministered after an interval of three weeks. Each child individually took the test. The complete item data are in Appendix G.

The Hoyt's reliability coefficient of the retention test was equal to .944; the standard error of measurement was 2.896.

This section presents some results from the item analyses for the terminal and retention tests, which are in the Appendices H and I, respectively.

Tables 6.2 and 6.3 show the difficulties for the items on the rules, identity (L), symmetric (S), rule K and neutron (N), for the following three forms: (1) a o b = ? (C), (2) ? o b = C (L), and (3) a o ? = c (R) for the Terminal Test and the Retention Test, respectively. The neutron rule is applied when the two elements a and b are the identity element ($\boxed{0\quad 0}$ lights 0). At a descriptive level it is apparent that the identity rule (I) was the most difficult.

TABLE 6.2

THE RULE ITEM AND MEAN DIFFICULTIES
ON THE TERMINAL TEST

Rule	Form	Difficulty	Mean
	C	.3750	
I	L	.4375	.4264
	R	.4667	
	C	.6416	
S	L	.4083	.4889
	R	.4167	
	C	.5208	
K	L	.5000	.5097
	R	.5083	
	C	.7500	
N	L	.7000	.7500
	R	.8000	

TABLE 6.3

THE RULE ITEM AND MEAN DIFFICULTIES ON THE RETENTION TEST

Rule	Form	Difficulty	Mean
I	C	.3504	.3860
	L	.4017	
	R	.4060	
S	C	.5812	.5128
	L	.4444	
	R	.5128	
K	C	.5128	.5470
	L	.5641	
	R	.5641	
N	C	.8205	.7607
	L	.7436	
	R	.7179	

Tables 6.4 and 6.5 show the average difficulty of the 16 items presented for each form in the terminal test and retention test, respectively. Although the differences are slight, the form L (? o b = c) was the most difficult in both tests.

TABLE 6.4

MEAN ITEM DIFFICULTIES OF THE FORMS ON THE TERMINAL TEST

Form	Difficulty
C	.5719
L	.5115
R	.5479

TABLE 6.5

MEAN ITEM DIFFICULTIES OF THE FORMS ON THE RETENTION TEST

Form	Difficulty
C	.5662
L	.5385
R	.5502

Conclusions Related to Empirical Proposition I

According the Empirical Proposition I, subjects who work in silence during physical mathematical learning activities perform better and retain more than subjects who verbalize overtly while learning. The proposition is the result of the fundamental principles and the forces of analysis and synthesis of the potential theory presented in Chapter II. Statistical analyses of the data for Hypotheses H_2 and H_5 indicate that both were rejected and therefore offer significant support for Proposition I.

It could be inferred that overt verbalization during questioning would increase performance and retention since, when subjects are asked to answer questions, they work in the symbolic plane and use social language. But statistical analyses of the null Hypotheses H_1 and H_4 indicate that it made no difference whether or not subjects talked during questioning. Language did not act as a shifter from the planes of perception and action to the plane of mental representation. That result could be interpreted as a consequence of the interference of the adult's language in the child's thinking. It may mean that the subjects did not understand the language used in the questions.

The fact that H_3 and H_6 were rejected gives considerable support for the strong interaction between learning and questioning suggested by the graphic interpretations in Figures 6.1 and 6.2, which represent terminal and retention test scores, respectively. Q, $\overline{Q}$, L, and $\overline{L}$ mean respectively: talk during questioning, no talk during questioning, talk during learning, and no talk during learning. These findings may explain the inconsistencies in the previous research on the issue of verbalization or nonverbalization in the learning of mathematics surveyed in Chapter III.

Conclusions Related to Empirical Proposition II

Empirical Proposition II says that if learning is done as established in Empirical Proposition I, then the process of discovering rules is accelerated. Statistical analyses of the Hypotheses H_7, H_8, H_9, H_{10}, and H_{11} give evidence for the validity of this second proposition. Children who did not talk during physical mathematical learning activities (learning from actions performed on objects) had a better opportunity to discover rules.

Conclusions Related to the Rules

Although analysis of data supports H_{12} and H_{13}, it rejects H_8, H_{10}, H_{14}, H_{15}, H_{16}, and H_{17}. This strongly suggests that the nature of all but the identity rule significantly affected scores on the terminal and retention tests. One explanation for the failure to reject H_{12} and H_{13} is that the subjects did not discover the identity rule. Analyses of their test scores presented in Appendices F and G

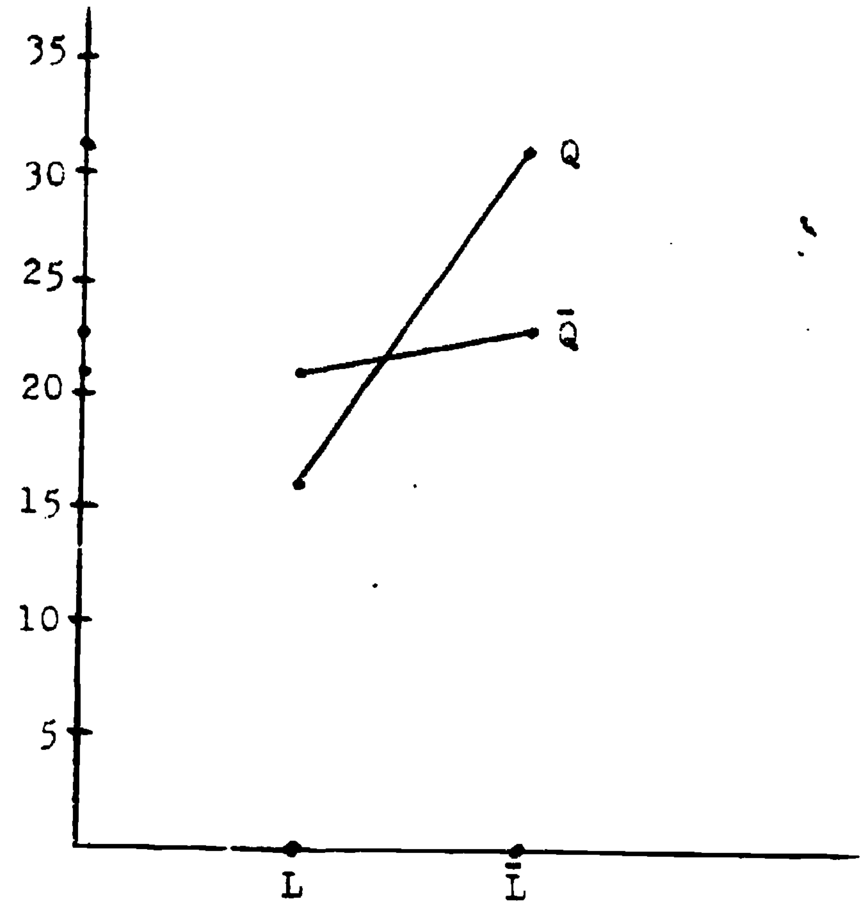

Figure 6.1. Interaction of talking during questioning (Q) and talking during learning (L) on the terminal test

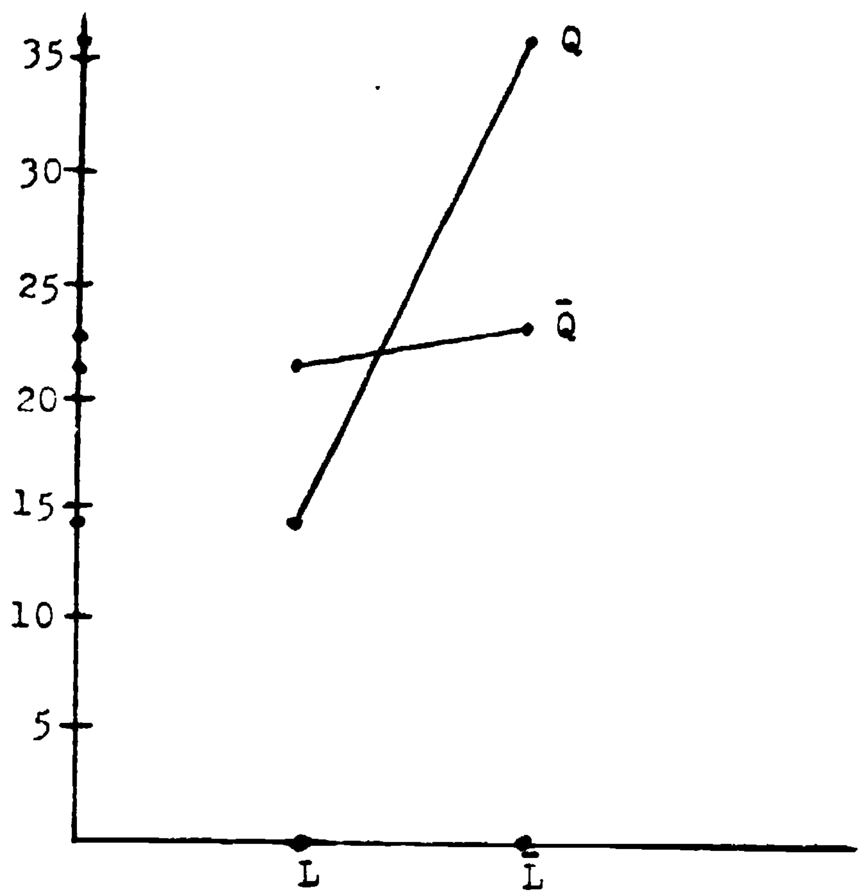

Figure 6.2. Interaction of talking during questioning (Q) and talking during learning (L) on the retention test

point to the identity rule as the most difficult to discover, the last of the rules to be acquired in the subjects' logical operational development.

Comparisons Among the Treatments

Comparisons among the means, using Scheffe's method, for the terminal test, retention test and number of rules show that the treatment $\overline{L}Q$ was the most effective. In the treatment $\overline{L}Q$ the subjects were silent while engaged in learning activities and afterwards answered questions and explained findings. This suggests positive influence of post-learning verbalization. Table 6.6 presents the descriptive data and the results of the Scheffe's analyses.

Conclusions Related to the Subjects' Strategies

The analyses of the subjects' strategies presented in Chapter V pointed out that the categories of strategies used for discovering rules could be observed with the consecutiveness method. The few subjects who discovered the identity rule used the synthetical method. The same subject could use different strategies for different rules, but the great majority who discovered rules used the synthetical strategy. This is evidence to support the assumption that when the subject discovers one rule, he tries to check it. Appendix J presents the graph analyses of the subjects' strategies.

TABLE 6.6

POSTERIORI COMPLEX COMPARISONS:
SCHEFFE'S METHOD

$$\psi = \bar{X}_2 - \frac{1}{3}(\bar{X}_1 + \bar{X}_3 + \bar{X}_4)$$

		Terminal Test (T.T.)		Retention Test (R.T.)		N. Rules (NR)	
		MS Error = 127.3056		MS Error = 104.8426		MS Error = 0.7889	
	Treat.	$\bar{X}_i$	No. Subjects	$\bar{X}_i$	No. Subjects	$\bar{X}_i$	No. Subjects
T1	(LQ)	17.3	10	14.9	10	0.4	10
T2	($\bar{L}$Q)	31.7	10	35.2	10	1.2	10
T3	(L$\bar{Q}$)	21.6	10	21.7	10	0.6	10
T4	($\bar{L}\bar{Q}$)	22.4	9	22.1	9	1.0	9
	F		22.4350		67.4042		8.1127
	$p <$		.001		.001		.001

Implications for Theory Construction

The findings of this study could suggest that building theory is not only respectable, but extremely useful, perhaps even indispensable in pursuing research on teaching. Snow (1973) comments that "the practice of researchers in past decades may be irrelevant when reviewed with improved models of the phenomena under study" (p. 77). The potential theory presented in Chapter II was the product of the author's ingenuity, but the findings of this study support it and justify the effort to develop and refine it further.

Implications for Curriculum Development

The findings of this study yield the following five tentative implications for mathematics curriculum development. However, the background of the population involved should be carefully considered before adopting any of these recommendations.

1. In instructional programs when learning from actions performed upon physical objects is intended, subjects should not verbalize.
2. In instructional programs with learning from actions performed upon physical objects, subjects should verbalize after completing physical learning activities in order to accelerate the process of learning.
3. The synthetical strategy for discovering rules should be encouraged as a powerful tool with which to develop logical operations.
4. In order to learn mathematical structures the explicit verbal use of rules should be discouraged.
5. In order to develop logical operations the use of the identity rule should be encouraged.

If these recommendations were followed more dynamic curricula could be built which would accelerate the learning of mathematics.

Recommendations for New Research

This study points out the advantage of basing a study upon a theoretic foundation. Such a foundation gave the author inspiration

and points the way toward more ingenious hypotheses from other researchers. An integrated plan of research may be necessary if theory building is to be maximized and if time consumption is to be minimized. The following studies are recommended as a starting point for such a integrated plan.

1. to repeat this study with other populations (across cultures, across sexes, and across ages with larger samples and with subjects selected randomly from these populations).
2. to conduct a similar study on the relationship between overt verbalization and answering questions using different mathematical learning activities. In particular, the differences between symbolic mathematical learning activities and physical mathematical learning activities should be investigated.
3. to replicate this study, but adding a transfer of learning task after the treatments.
4. to conduct further research on the relationship between the nature of the rules to be discovered and performance.
5. to conduct further research on the relationship between the identity rule and performance. Such studies should be done across ages, across sexes and for subjects with different levels of intelligence and with different logical operations.
6. to design studies to examine the relationship between the identity rule and the development of logical operations.

This study was conducted to examine two theorems derived from a potential theory. From it the author gained experience in the art of inquiry.

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Appendix A

GENERAL ACHIEVEMENT TEST

NAME ______________________________

SCHOOL: ______________________________

GRADE : ________________

DATE OF BIRTH : __________ - __________ - __________

(Month) (Day) (Year)

SEX : MALE ☐

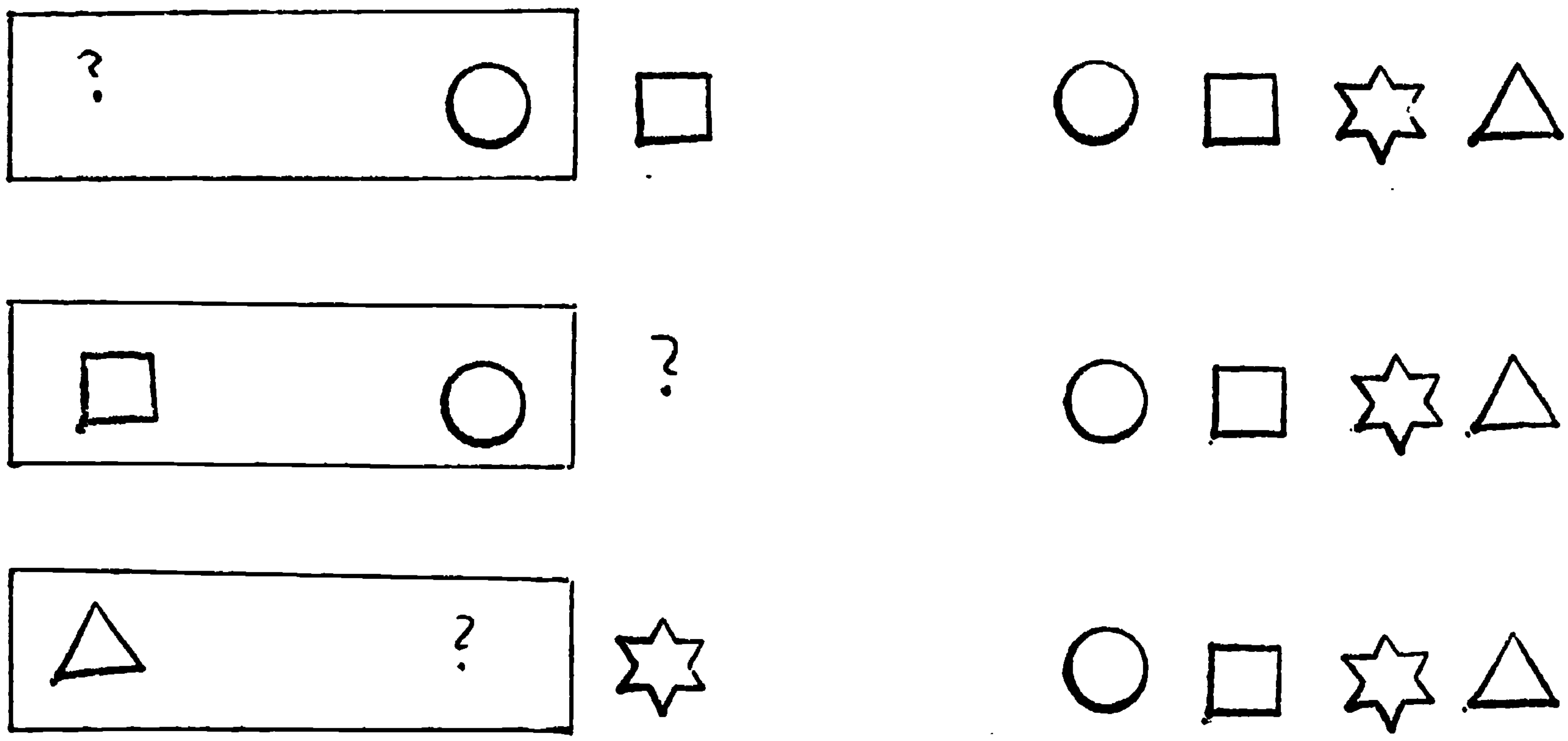
FEMALE ☐

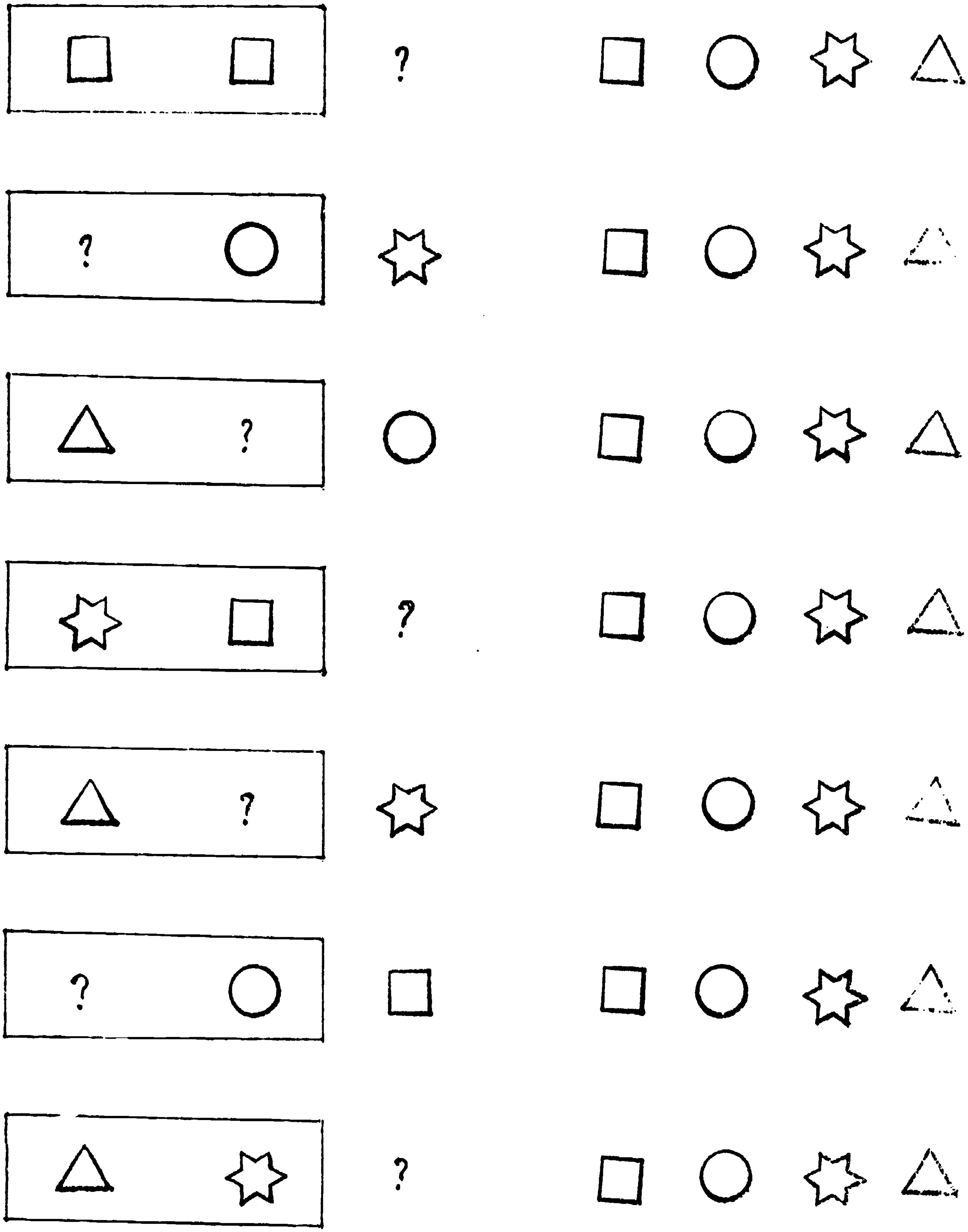
DATE : ________________

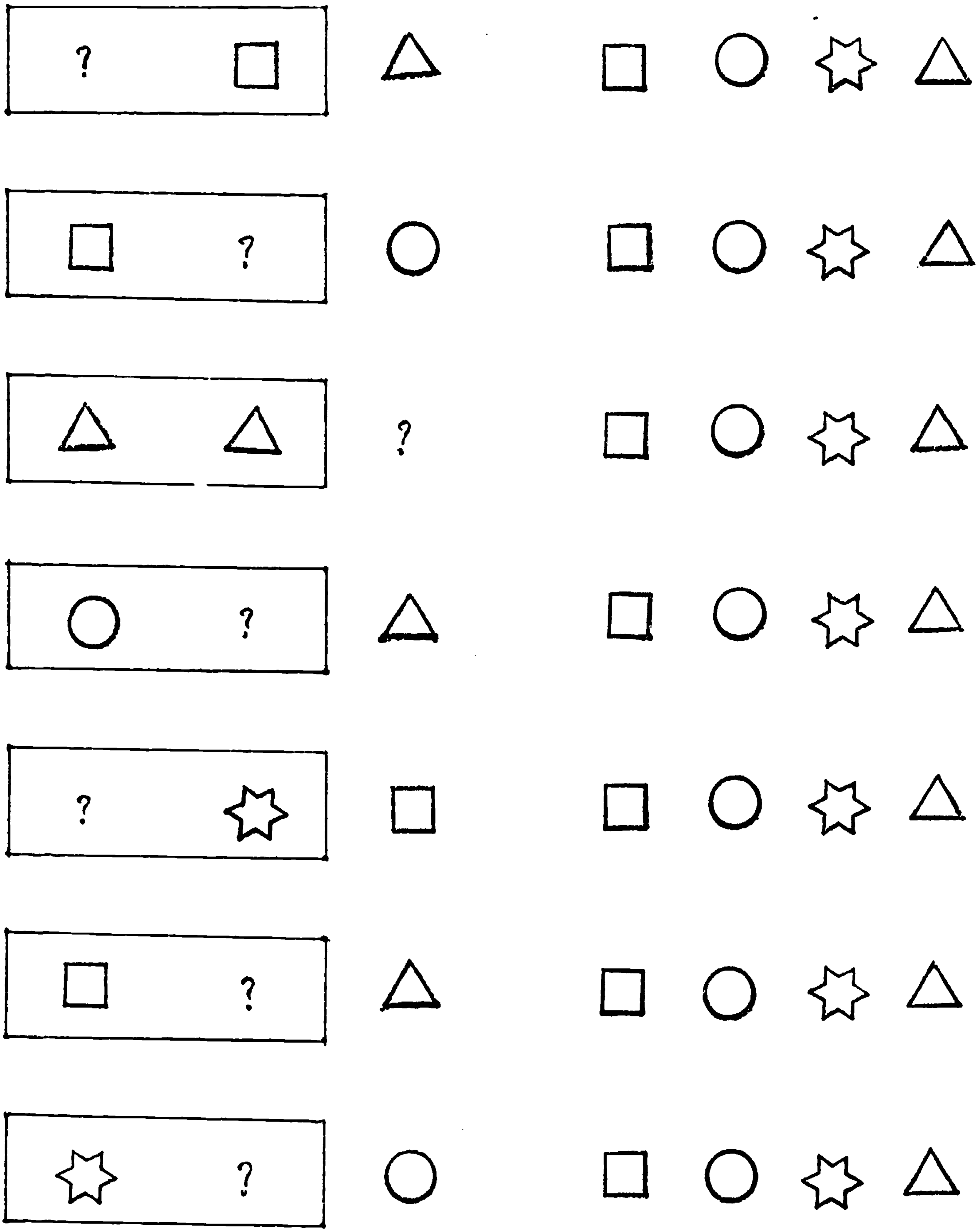
EXERCISES

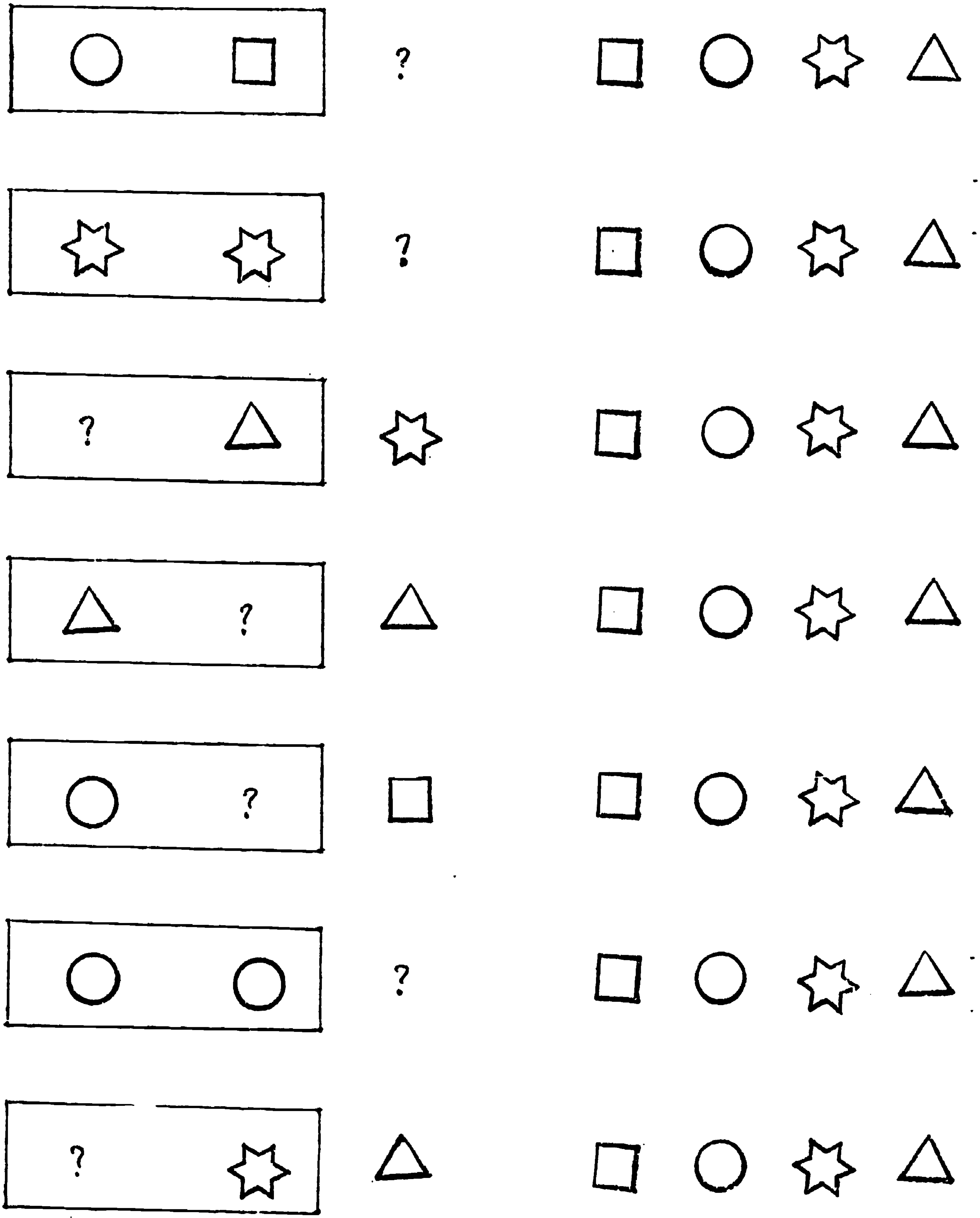
EXAMPLES

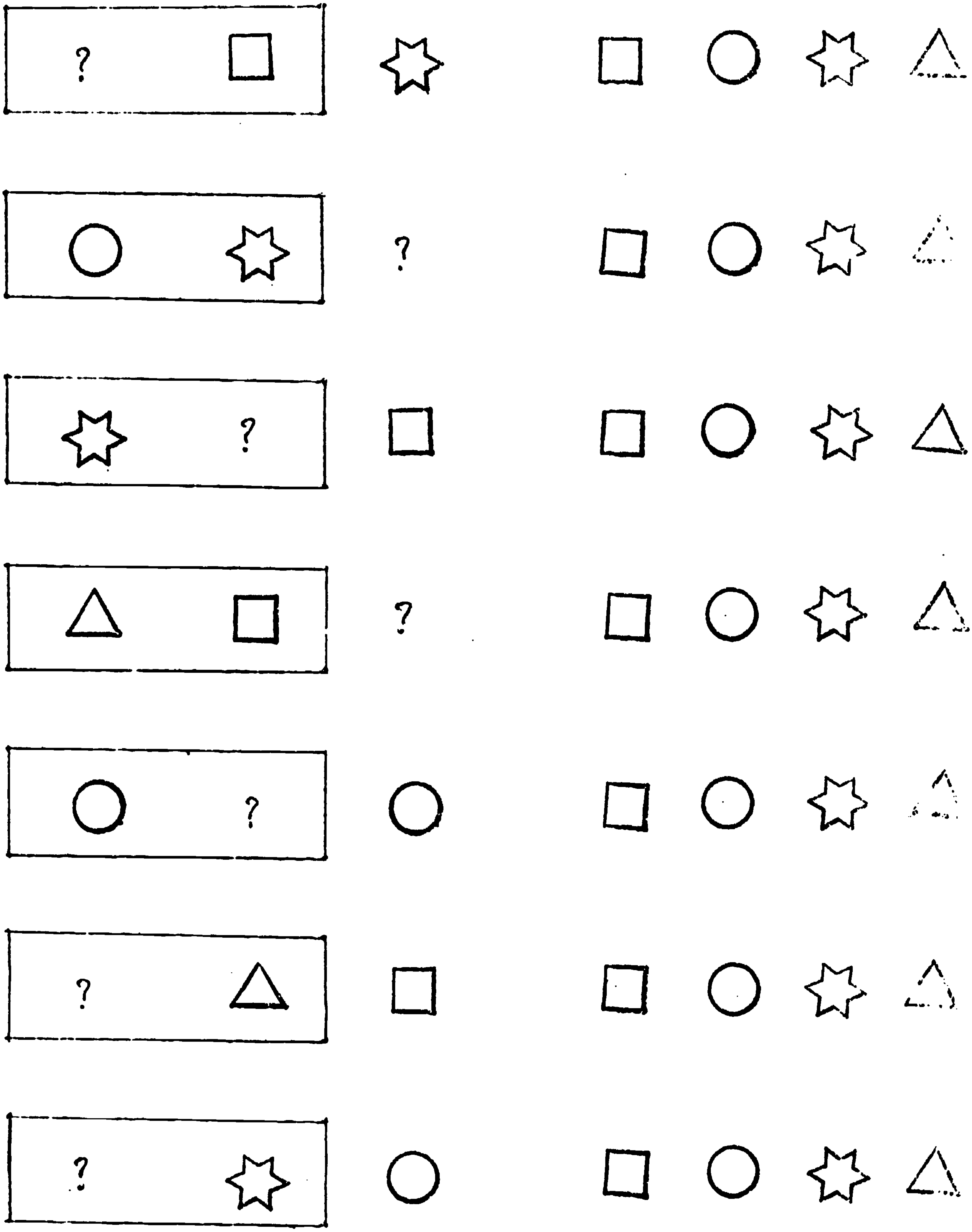
THE PAIR OF FIGURES	THE FIGURE LIGHTNED BY THE MACHINE	MARK THE FIGURE THAT YOU PREDICTED

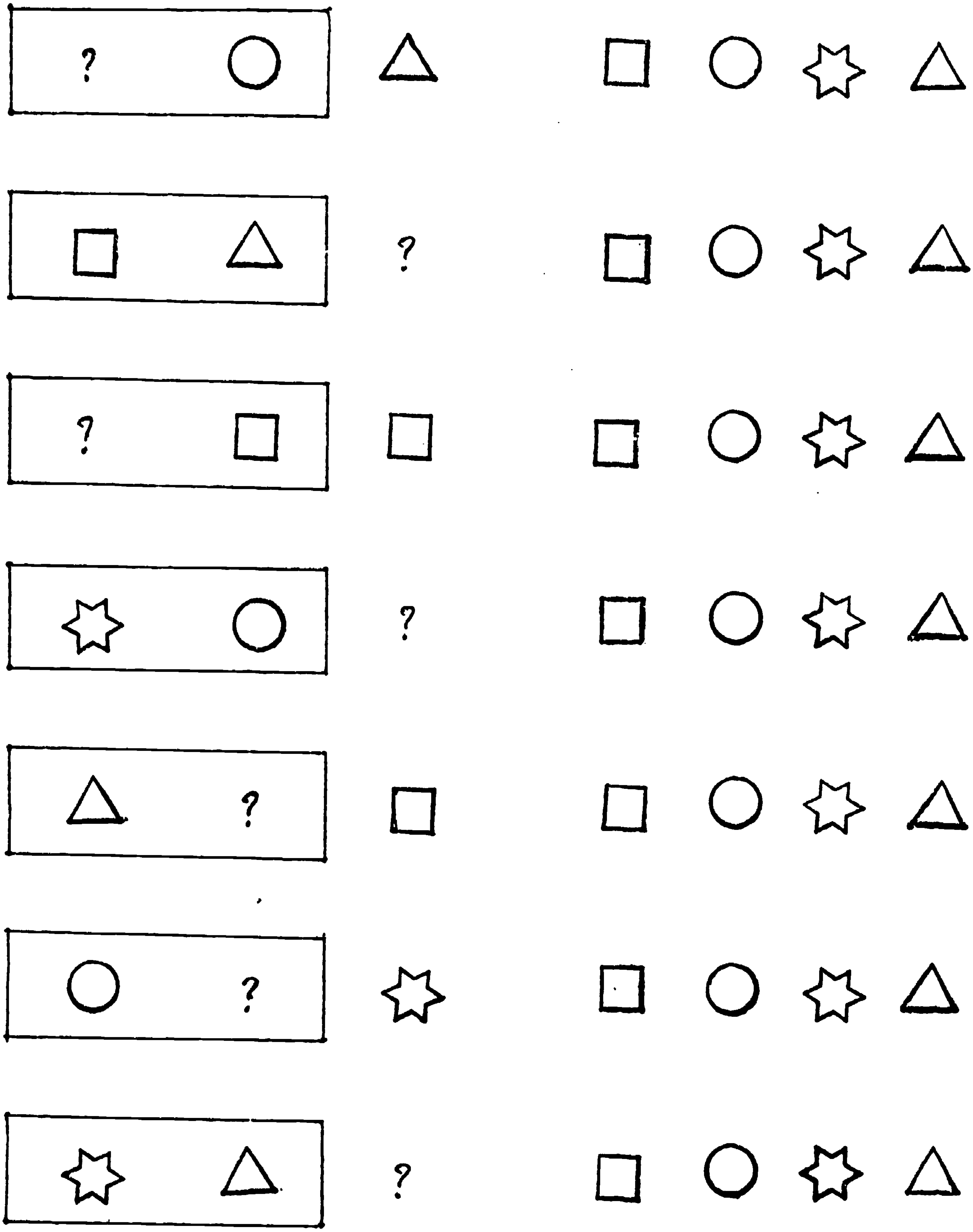


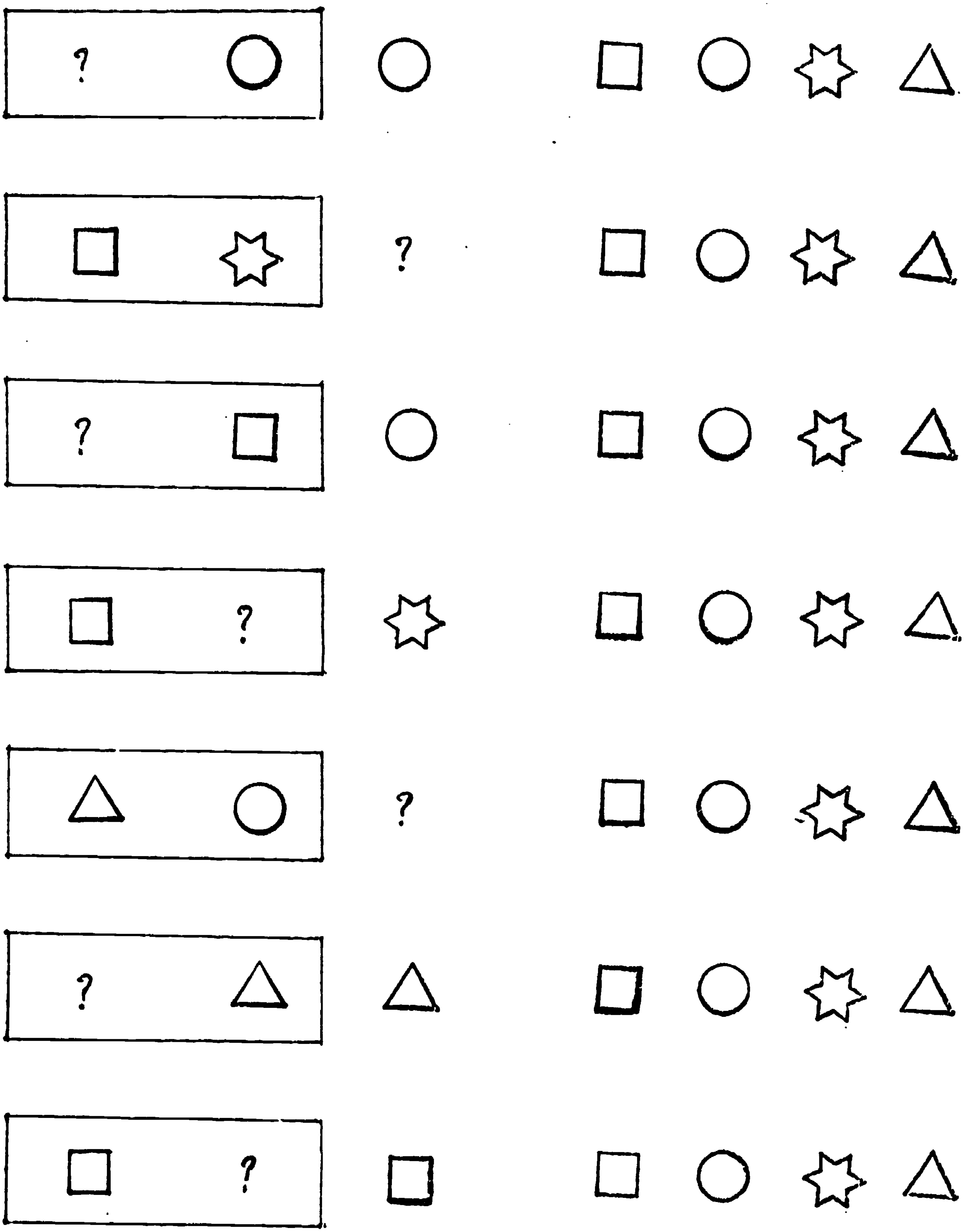


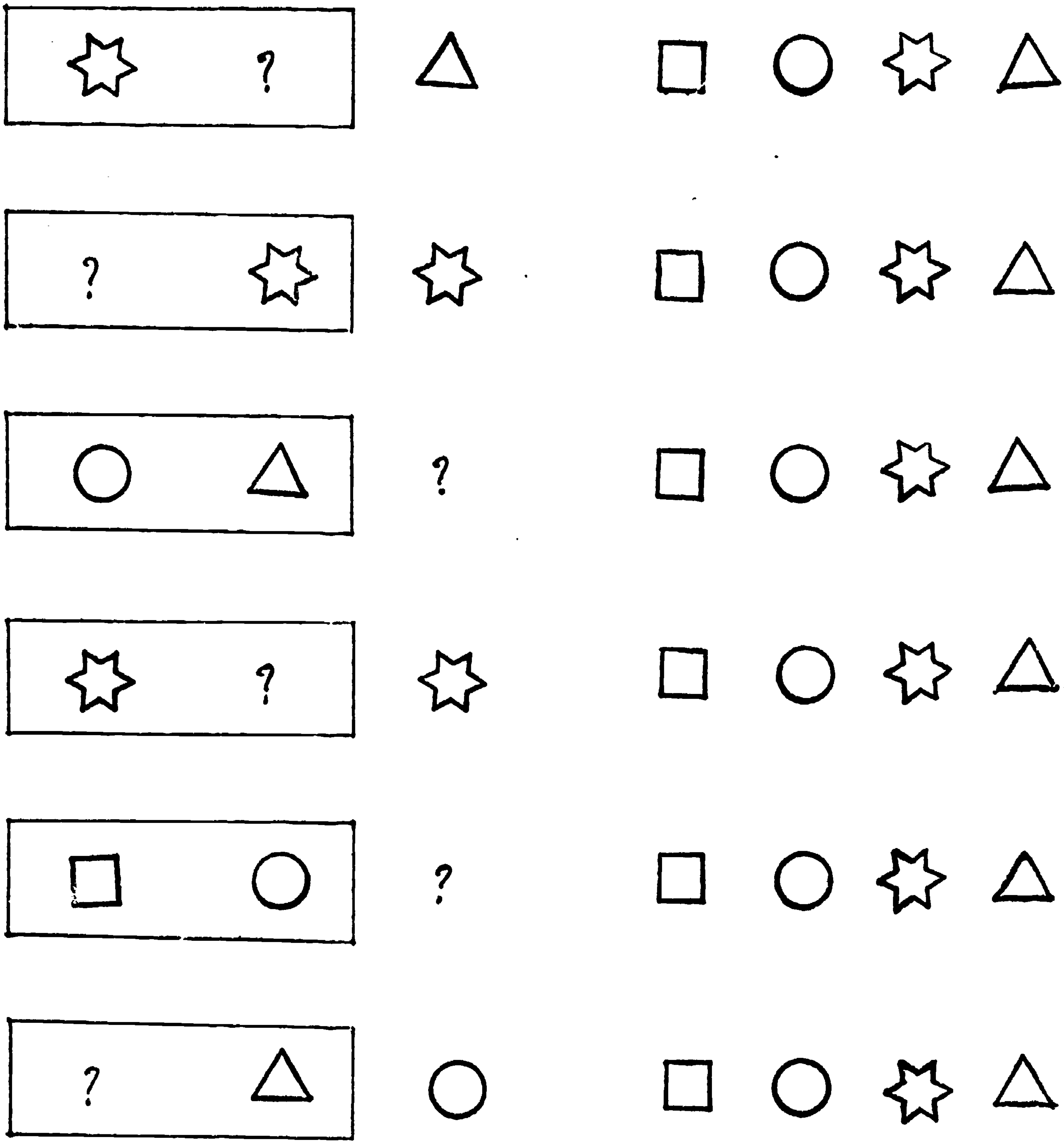












Appendix B

LORGE-THORNDIKE TEST

INSTRUCTIONS

Print your name on the cover of this booklet.

This booklet contains two short sets of questions. You are to mark your answers by circling your choice.

All questions are followed by five choices, only one of which is the right answer.

Part A

INSTRUCTIONS

Look at sample question 0.

0. rose daisy violet

A red B garden C sweet D grow E lily

The words in question 0 are the names of flowers. On the next line only lily is the name of a flower. The letter before lily is E, so E has been circled.

Now look at question 00. Think in what way the words in question 00 go together. Then find the word on the line below that belongs with them.

00. go run walk move

F think G dream H march J sing K seem

The right answer is march. Circle the H answer.

Do all of the questions on the next two pages in the same way. Try every question. Mark only one answer for each question.

Wait for the signal to begin.

1. bench seat stool
   A table B chair C desk D bed E sit

2. potato beet pea
   F nut G banana H vegetable J dinner K carrot

3. book magazine letter
   L movie M newspaper N radio P lecture Q read

4. sheep pig cow horse
   R dog S rabbit T deer U wolf V beaver

5. peel rind bark shell
   A corn B orange C tree D husk E box

6. dollar peso mark lira
   F change G franc H foreign J purchase K bank

7. musician actor humorist singer
   L ventriloquist M professional N amateur P program Q radio

8. alley road drive path
   R country S glade T passageway U glen V lane

9. stairway ladder stairs staircase
   A elevator B climb C hill D escalator E grade

10. herd flock swarm drove
    F lair G den H bunch J pack K insects

11. car cab wagon cart
    L train M carriage N vehicle P motor Q tandem

12. pin safety pin hook and eye zipper
    R button S belt T strap U suspenders V garters

13. tie cravat stock neckcloth
A bib B collar C scarf D kirtle E girdle

14. honesty loyalty sincerity faithfulness
F passivity G servility H devotion J obsequiousness K compliance

15. pine spruce hemlock
L chestnut M willow N poplar P fir Q maple

16. caricature parody burlesque satire
R reflect S echo T parrot U simulate V mimic

17. emerald lawn leaf spinach
A diamond B mower C shamrock D bean E stamp

18. gavotte waltz polka one-step
F ballet G masquerade H ball J orchestra K minuet

19. house skyscraper hospital museum
L library M store N railroad P office Q fort

20. accidental fortuitous random casual
R unessential S extrinsic T extraneous U accessory V chance

21. town city capital metropolis
A province B county C suburb D esplanade E country

22. aviary apiary menagerie hatchery
F incubator G hive H garden J aquarium K warren

23. shutter lens film filter
L diaphragm M camera N negative P print Q exposure

24. bottle lens window spectacles
R vase S electric bulb T plaque U lamp V dish

25. furtive stealthy clandestine secretive
A reserved B surreptitious C cryptic D private E mystic

STOP! Wait for further directions.

Part B

INSTRUCTIONS

Look at sample question 0.

0. laugh → happy : cry →

A wonder B sad C hide D lost E rough

The right answer is B because sad belongs with cry just as happy belongs with laugh. We have circled the B answer.

Now look at question 00.

00. chair → sit : bed →

F lie G bedroom H night J crib K tired

The right answer is F because lie belongs with bed just as sit belongs with chair. Circle the F answer.

Wait for the signal to begin. Do all the questions on the next page in the same way. Try every question.

Part B

1. forest → tree : garden →
   A rake B gladiolus C blossom D flower E fruit
2. book → chapter : play →
   F stage G scenery H cast J act K drama
3. handkerchief → linen : dress →
   L dressmaker M cotton N style P apparel Q print
4. automobile → manufacture : home →
   R rent S buy T build U mortgage V own
5. human being → arm : tree →
   A trunk B twig C limb D foliage E growth
6. biology → microscope : astronomy →
   F telescope G binoculars H lens J stratosphere K heavens
7. speaker → introduction : author →
   L contents M index N digest P title page Q preface
8. laborer → wage : teacher →
   R profession S work T fee U honorarium V salary
9. plaintiff → defendant : prosecution →
   A litigation B decision C defence D replication E appellant
10. king → abdicate : president →
    F disdain G retract H resign J veto K coup d'état
11. federal → congress : state →
    L house M senate N representatives P constitution Q legislature
12. jeopardy → security : hazard →
    R quarantine S safeguard T custodian U peril V convoy
13. distill → extract : precipitate →
    A deposit B colloid C solidify D congeal E isotope
14. diffuseness → expansion : conciseness →
    F terseness G condensation H laconicism J epithet K ellipsis
15. vindicate → acquit . stigmatize →
    L prosecute M libel N arraign P condemn Q indict

STOP! Wait for further directions.

Appendix C

CODES

INPUT DATA

1, 2..............	Subject number		
3	Treatments	1	LQ (talk, talk)
		2	$\bar{L}Q$ (no talk, talk)
		3	$L\bar{Q}$ (talk, no talk)
		4	$\bar{L}\bar{Q}$ (no talk, no talk)
4, 5, 6	Age in months		
7, 8, 9	I.Q.		
10, 11	Part A (Verbal Test)		
12, 13	Part B (Verbal Test)		
14, 15	Terminal Test (T.T.) (wrong answers)		
16, 17	Terminal Test (T.T.) (right answers)		
18, 19	Retention Test (R.T.) (wrong answers)		
20, 21	Retention Test (R.T.) (right answers)		
22, 23,.......69	Codes of the choices and predictions of the subjects during learning.(Rules)		

RULES	CODES
a. Identity	1- Right
	4- Wrong
b. Symmetric	2- Right
	5- Wrong

c. Rule K .... 3- Right

6- Wrong

NOTE

(circle, circle was coded in the following way:

(i) The nature of the closer choice (identity or symmetric) gave the nature of the choice (circle, circle).

(ii) In the case of same distance, the nature of the precedent choice gave the nature of the choice (circle, circle) .

Appendix D

INPUT DATA

INPUT DATA

```
0131390840905311734146132462562446161266143461636253156656646465451 22
022137082080429191632164446266462246642134641616351262626646266213616
031148080080532163117524114664165256464255366426441344614652263426141
044147          3909     33455446444661165654444126321642266414646411 6562
054140078050543053612653546461466164655546654616665315521444616411164
061136089110627213315466452646512454266652446616664464646461664461446
073132101151122260840561521564416625166664646446645252166616664631664
082132072010136123612644666414511661544466563454242641611511624561 4662
092138081020925233216265646164465446541556666534653466613415211466546
101146089120932162820551234444463533424435644613446211626234134121562
114133095130910381632113342146346234364432234411263333114131322321233
123133081050344043909645464344564635536644642265644346446544656462426
133135082040636123414465343461465446523644624465356441663556661 1.11526
141135092140630183513445325465464554646514433444414541465555115116255
154141092151130182919541321465145314246132664141466254164331656325465
162140093121105430642412533424611654162235646441433344412223144411431
171140088120729193216563536345444146655541166463113544532311656316415
183139092130820281434133464554641165416643656651421446633613544116643
194140084070834143018114661563346413456456436611264431341656333523553
202140092150706420246135566454616352351566442654456613611532354613366
214135102160922262820644645353466143445252114166411624636413464122646
222131086080537112325346455644344461566464233313454432344344264344464
233133095170534143315265161664462416526643323564541415664546453411346
241138094140930183018661446565554561616643344543623611664135636461164
254131083100517312028414635555466242126263242662661223312334632242121
262140091120908400048341436533662464323144162213311613616632313423213
271150074050429193612146252244543643444543436346414656631346353435254
283139094131018301533154166651126454661656411641465136463544355233664
293140081070529194107416565414442552664612646156443655546356535655161
```

```
3041370911307351332162646266451162656116164446514623234411646146544662
3121350891008044402461446644131523132133133131331113133131321313333331
3241410982007014708404615652543346523113131135113522261332311344222222
3311370850708371137114611645454646466126644162116644466236434565114324
3421400881306133510385436346444466345356543141414531532414133334213 23
3531410750502054330184265465355633515465463641251646612616236261161 61
3611400921705361234143625545261346652664313514636664154153254665234 54
3711310801006252335132443355466445144266213615534525236442544225342 21
383138089120725231533116646413336355221144366336363333411634114634331
3941290880607252334141616655143541116532366321535344435514165353413 21
402132099160800480147266163111624661421132323113161124132323113131432
```

(MADISON ACADEMIC COMPUTING CENTER PROGRAM) 73/10/09

Appendix E

ANALYSES OF THE RULES

(OUTPUT OF THE COMPUTER)

I.D.	Treat.	$CONS \vee \overline{CONS}$ I	$CONS \vee \overline{CONS}$ S	$CONS \vee \overline{CONS}$ K	RATIO I	RATIO S	RATIO K	TT	RET	SCORE
1	3	0	1	0	0	1	0	17	14	82.4
2	2	0	1	0	0	1	0	19	32	168.4
3	1	0	1	0	0	1	0	16	17	106.3
4	4	1	0	0	0	0	0	9		
5	4	0	0	0	0	0	0	5	12	240.0
6	1	0	0	0	0	0	0	21	15	71.4
7	3	0	0	0	0	0	0	26	40	153.8
8	2	0	0	0	0	0	0	12	12	100.0
9	2	0	0	0	0	0	0	23	16	69.6
10	1	0	0	0	0	1	1	16	20	125.0
11	4	1	1	1	1	1	1	38	32	84.2
12	3	0	0	0	0	0	0	4	9	225.0
13	3	1	0	0	0	0	0	12	14	116.7
14	1	1	0	0	0	0	0	18	13	72.2
15	4	0	0	0	0	0	0	18	19	105.6
16	2	0	1	1	0	1	1	43	42	97.7
17	1	0	0	0	0	0	0	19	16	84.2
18	3	0	0	0	0	0	0	28	34	121.4
19	4	0	0	1	0	0	0	14	18	128.6
20	2	0	0	0	0	0	0	42	46	109.5
21	4	0	1	0	0	1	0	26	20	76.9
22	2	0	0	0	0	0	1	11	25	227.3
23	3	0	0	0	0	0	0	14	15	107.1
24	1	0	0	0	0	0	0	18	18	100.0

25	4	0	1	0	0	1	0	31	28	90.3
26	2	0	1	1	1	1	1	40	48	120.0
27	1	0	0	1	0	0	1	19	12	63.2
28	3	0	0	0	0	0	0	30	33	110.0
29	3	0	0	0	0	0	0	19	7	36.8
30	4	0	0	0	0	1	0	13	16	123.1
31	2	1	1	1	1	1	1	44	46	104.5
32	4	0	1	1	1	1	1	47	40	85.1
33	1	0	0	0	0	0	0	11	11	100.0
34	2	0	1	1	0	0	1	35	38	108.6
35	3	1	1	0	1	0	0	43	18	41.9
36	1	0	0	0	0	0	0	12	14	116.7
37	1	0	1	0	0	1	0	23	13	56.5
38	3	0	1	1	1	0	1	23	33	143.5
39	4	0	0	1	1	0	1	23	14	60.9
40	2	0	1	1	1	1	1	48	47	97.9

Appendix F

INPUT DATA--TERMINAL TEST

```
0110001010000110001100110111000010001010001001 0000
021000000100101010101011001110000111001110000000001
030001001001011001110110001100000010110010000000000
0401[illegible]0000001000000000011000110001011000000000000000
05000001000000001000110010010100000101000000000001
061010110001000101010100100100000101010110110111 00
071010101010010001110111001100011010111110011000 11
081011010011010000010110010101000101010000010000000
090100111000111000101111011110010100111010001000 00
101000101001000000000101001100001010011010011000 10
111111111011010101101111111111111110011111011101101
120001000000000000100000000000000000010000000010000
130000000001110100000001000001001000000000000110010 11
14000000111011001000001010111100100001100100110000 0
150001101000101101110000011101000000011001010000 10
161101111111111111111111111111100111101110111111111
171100010000100110001110100101000001011100000101 10
181000101001011011011011001010011101001001111111 11
190001011000000000100111010100010000011000000100001
200101111110111111111111111111111111010110111111111
211010010111001111001101000100001001010011111111110
221011001001010001000000000000100001000110000000000
230100010000110000000101100100011100010001000000100
240000011101000011001011010000111000000010110000 11
251100011001111011101111111011000100101011011011111
261111110111100111111101100101111111111101111111111
271000100111001000100110001110010000101110011000 00
281010110111001111111110011011011101111010000000 11
291001110000000101001101001001001101000101001100100
300000010100100000101001100100000000000001001100101 00
311111111110111111111111111111110111111111101011111111
321111111110111111111111111111111111111111111111111111
331000100001001000000001000001001101000000100000 1000
341011101111010101110111011111011010111110111101 01
351101111111111111111111110111111111111111111111010101
361010100000010001000011000010000001001000000101 00000
371011000011001101001111000111000110110101000001 01
381000010111011111000101010100000000101111000110110
390110110001100111000100001100011010111010010001 11
401111111111111111111111111111111111111111111111111
```

Appendix G

INPUT DATA--RETENTION TEST

01 10000000000011001110111001110000000000000011000000
02 111001001111011101110010010110111101011101011111
03 010110110000100010011000111000001001101000100000
04 --
05 0010000000000001000110010011000010101000000001001
06 100000001100001100010100111000000101000000001011
07 1110101111011111111111011111111101011111011111101
08 0000110010000000001100000001001000010110000100010
09 001001000010010100010000100101000011111000100010
10 100110111011000001011100011000100011100001000010
11 1101101011000101100111111111101010101111100011111
12 00001000010000100001000100010000001000101000000000
13 1010110000000000000001001110100011001000011000000
14 00011010000000000010110100010010010001001000100000
15 110110011000000101010100010111000000110001010110
16 110111111110100111110111111111111101111111111111
17 001001010010101000111000010000010001101000100010
18 100101101111110111111101001011101001101011111111
19 000110010001100011011100111000000001100110100000
20 011011111111111111111111111111111111111111111111
21 1000001011100010110110101001011001000001110001000
22 101110111100110110011101011001110001001001000001
23 000010110000010010010000010000010001111101001000
24 000111010001101000100101100001101010011001000000
25 101001101101111110111000011101101011110011000001
26 111111111111111111111111111111111111111111111111
27 1000101100001000100010000000001001010000001100000
28 10101111110111011111111001111011111001101001100011
29 100010010000100000000010010000000100010101000110010
30 0000000011101100101011001000001101001000001010000
31 111111111111111111111111111111111111111101011111111
32 0111110110111110111111111111111111111111111110000
33 10000010000011000000000001010010000000100010001100
34 101110111001111111111101111101001011111111111011
35 1000110101111101100111000100000000001101000000000
36 10101010000000001000101010100010000011000000010001
37 1010000011001000000010000010000101001010010000001
38 10111111111111111111110001010101101001010011100111
39 000010001100000000101000101000100010100001110010
40 111111111111111111111111111111111111111110111111111

Appendix H

ITEM ANALYSES - TERMINAL TEST

ITEM ANALYSES
TERMINAL TEST

ITEM	NCR	DIFF.	R dis	X50	BETA	RULE	FORM
1	26.	.6500	.4217	-.7096	.6466	S	C
2	14.	.3500	.5992	.4994	1.2132	I	L
3	15.	.3750	.3870	.6449	.5684	S	R
4	17.	.4250	.4390	.3415	.6651	K	C
5	23.	.5750	.4947	-.3031	.7986	K	R
6	23.	.5750	.4732	-.3168	.7442	I	L
7	22.	.5500	.3873	-.2582	.5572	K	C
8	14.	.3500	.6525	.4586	1.5496	K	L
9	16.	.4000	.5877	.3399	1.1178	S	R
10	26.	.6500	.2441	-1.2257	.3312	S	C
11	17.	.4250	.3705	.4047	.5286	I	R
12	19.	.4750	.5024	.0995	.8115	K	L
13	20.	.5000	.3938	.0000	.5675	K	R
14	19.	.4750	.6508	.0768	1.4128	S	R
15	18.	.4500	.5234	.1910	.8736	I	C
16	25.	.6250	.5357	-.4659	.9375	S	C
17	18.	.4500	.6426	.1556	1.3699	K	L
18	19.	.4750	.5066	.0987	.8229	I	R
19	18.	.4500	.6341	.1577	1.3196	I	R
20	30.	.7500	.4254	-1.1637	.7113	N	C
21	27.	.6750	.3142	-1.1099	.4480	K	L
22	24.	.6000	.4970	-.4020	.8117	K	L
23	12.	.3000	.5313	.7489	.9809	I	C
24	20.	.5000	.4954	.0000	.7921	K	R
25	21.	.5250	.4770	-.1048	.7465	K	C
26	32.	.8000	.2858	-2.0610	.4474	N	R
27	18.	.4500	.5447	.1835	.9394	K	L
28	14.	.3500	.4084	.7327	.6182	S	L
29	9.	.2250	.6109	.8881	1.6180	I	L
30	21.	.5250	.5490	-.0911	.9496	K	C
31	23.	.5750	.5418	-.2767	.9362	I	L
32	17.	.4250	.5675	.2642	1.0252	I	C
33	17.	.4250	.3148	.4763	.4327	K	R
34	16.	.4000	.4408	.4532	.6741	I	R
35	19.	.4750	.4346	.1151	.6501	K	C
36	28.	.7000	.5036	-.7901	.8873	N	L
37	25.	.6250	.4832	-.5165	.7839	K	C
38	15.	.3750	.3958	.6306	.5855	S	L
39	23.	.5750	.5118	-.2929	.8454	K	R
40	12.	.3000	.4759	.8361	.8052	I	C
41	19.	.4750	.5448	.0918	.9358	I	L
42	22.	.5500	.5873	-.1702	1.0942	I	R

ITEM ANALYSES
TERMINAL TEST

ITEM	NCR	DIFF.	R dis	X50	BETA	RULE	FORM
43	19.	.4750	.4854	.1030	.7675	K	R
44	17.	.4250	.5332	.2812	.9090	I	L
45	13.	.3250	.5673	.6147	1.0944	I	C
46	20.	.5000	.6351	.0000	1.3152	I	R
47	18.	.4500	.4724	.2116	.7379	I	C
48	20.	.5000	.6097	.0000	1.1848	S	L

Appendix I

ITEM ANALYSES - RETENTION TEST

ITEM ANALYSES
RETENTION TEST

ITEM	NCR	DIFF.	R dis	X50	BETA	RULE	FORM
1	25.	.6410	.3053	-.9217	.4260	S	C
2	11.	.2821	.6977	.6206	2.5177	I	L
3	18.	.4615	.4829	.1593	.7622	S	R
4	16.	.4103	.4782	.3750	.7598	K	C
5	27.	.6923	.2610	-1.4668	.3646	K	R
6	18.	.4615	.4913	.1565	.7836	I	L
7	20.	.5128	.5052	-.0508	.8183	K	C
8	23.	.5897	.4105	-.4369	.6077	K	L
9	23.	.5897	.6370	-.2816	1.3603	S	R
10	20.	.5128	.4926	-.0521	.7850	S	C
11	13.	.3333	.5796	.5732	1.1388	I	R
12	19.	.4872	.5544	.0463	.9664	K	L
13	23.	.5897	.4618	-.3884	.7198	K	R
14	19.	.4872	.6385	.0402	1.3349	S	R
15	14.	.3590	.4219	.6670	.6442	I	C
16	23.	.5897	.6498	-.2760	1.4436	S	C
17	22.	.5641	.5839	-.2195	1.0849	K	L
18	18.	.4615	.6431	.1196	1.3682	I	R
19	20.	.5128	.5809	-.0441	1.0624	I	R
20	32.	.8205	.3935	-1.5913	.7055	N	C
21	23.	.5897	.3592	-.4992	.5102	K	L
22	22.	.5641	.5076	-.2525	.8312	K	L
23.	9.	.2308	.6573	.8089	2.1993	I	C
24	19.	.4872	.5249	.0488	.8739	K	R
25	19.	.4872	.4156	.0617	.6104	K	C
26	28.	.7179	.3999	-1.0828	.6293	N	R
27	23.	.5897	.4575	-.3920	.7098	K	L
28	17.	.4359	.6833	.1875	1.6891	S	L
29	12.	.3077	.6543	.5850	1.6764	I	L
30	23.	.5897	.5387	-.3329	.9312	K	C
31	19.	.4872	.5796	.0442	1.0574	I	L
32	15.	.3846	.4562	.5051	.7136	I	C
33	20.	.5128	.6019	-.0426	1.1497	K	R
34	9.	.2308	.5226	1.0173	1.0489	I	R
35	17.	.4359	.3655	.3507	.5184	K	C
36	29.	.7436	.4521	-1.0677	.7758	N	L
37	25.	.6410	.3754	-.7496	.5499	K	C
38	17.	.4359	.4969	.2579	.6021	S	L
39	23.	.5897	.4831	-.3712	.7723	K	R
40	10.	.2564	.4528	1.0660	.7778	I	C
41	18.	.4615	.4829	.1593	.7622	I	L
42	23.	.5897	.5558	-.3227	.9888	I	R

ITEM ANALYSES
RETENTION TEST

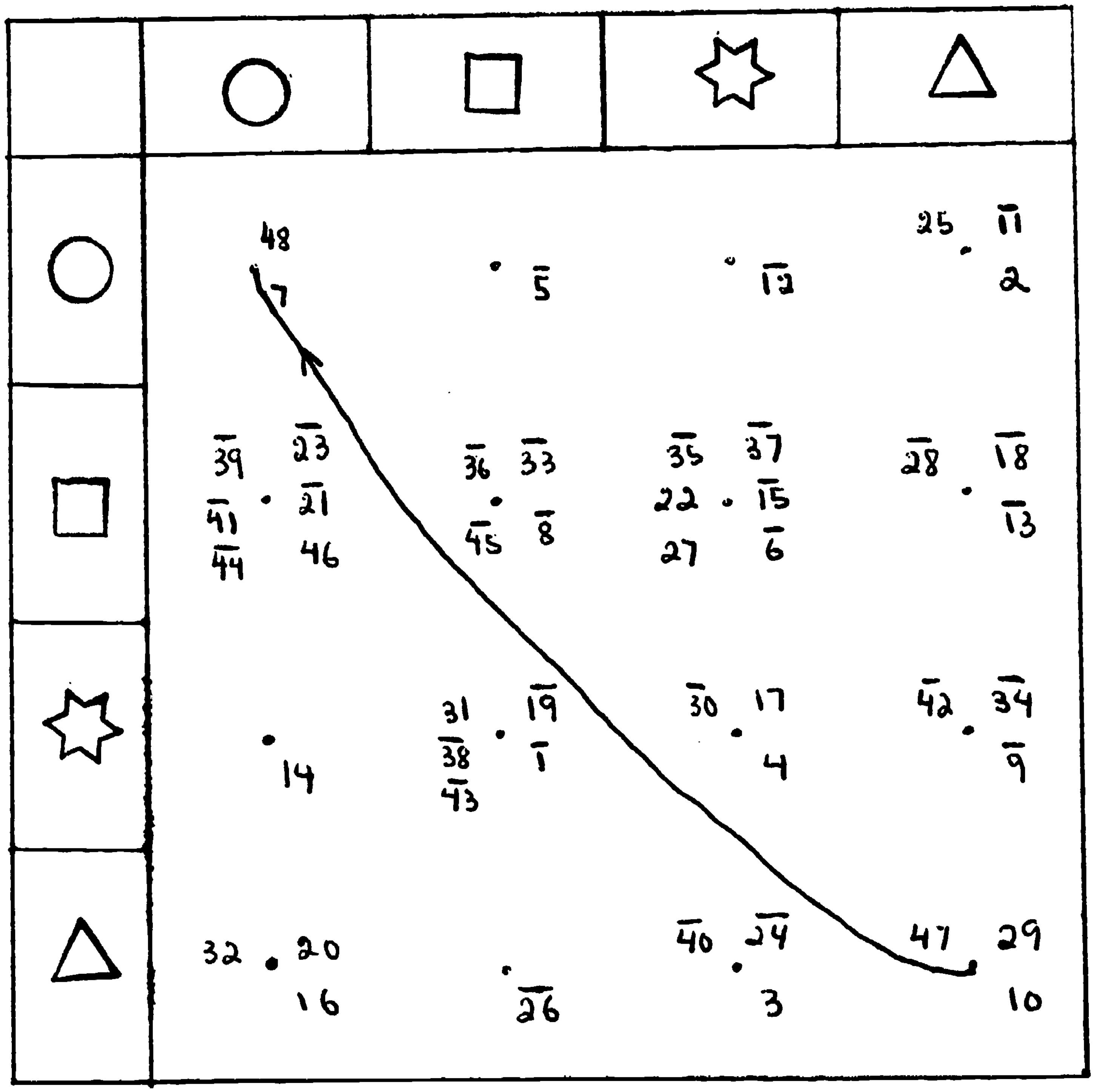
ITEM	NCR	DIFF.	R dis	X50	BETA	RULE	FORM
43	20.	.5128	.4379	-.0585	.6568	K	R
44	16.	.4103	.6107	.2937	1.2167	I	L
45	15.	.3846	.5297	.4351	.9132	I	C
46	12.	.3077	.6680	.5730	1.8228	I	R
47	19.	.4872	.4493	.0571	.6815	I	C
48	18.	.4615	.6389	.1204	1.3429	S	L

Appendix J

GRAPH ANALYSES

0 1 3

GRAPH I


48
7
$\overline{5}$
$\overline{12}$
25 $\overline{11}$
2
$\overline{39}$ $\overline{23}$
$\overline{41}$ $\overline{21}$
$\overline{44}$ 46
$\overline{36}$ $\overline{33}$
$\overline{45}$ $\overline{8}$
$\overline{35}$ $\overline{37}$
22 $\overline{15}$
27 $\overline{6}$
$\overline{28}$ $\overline{18}$
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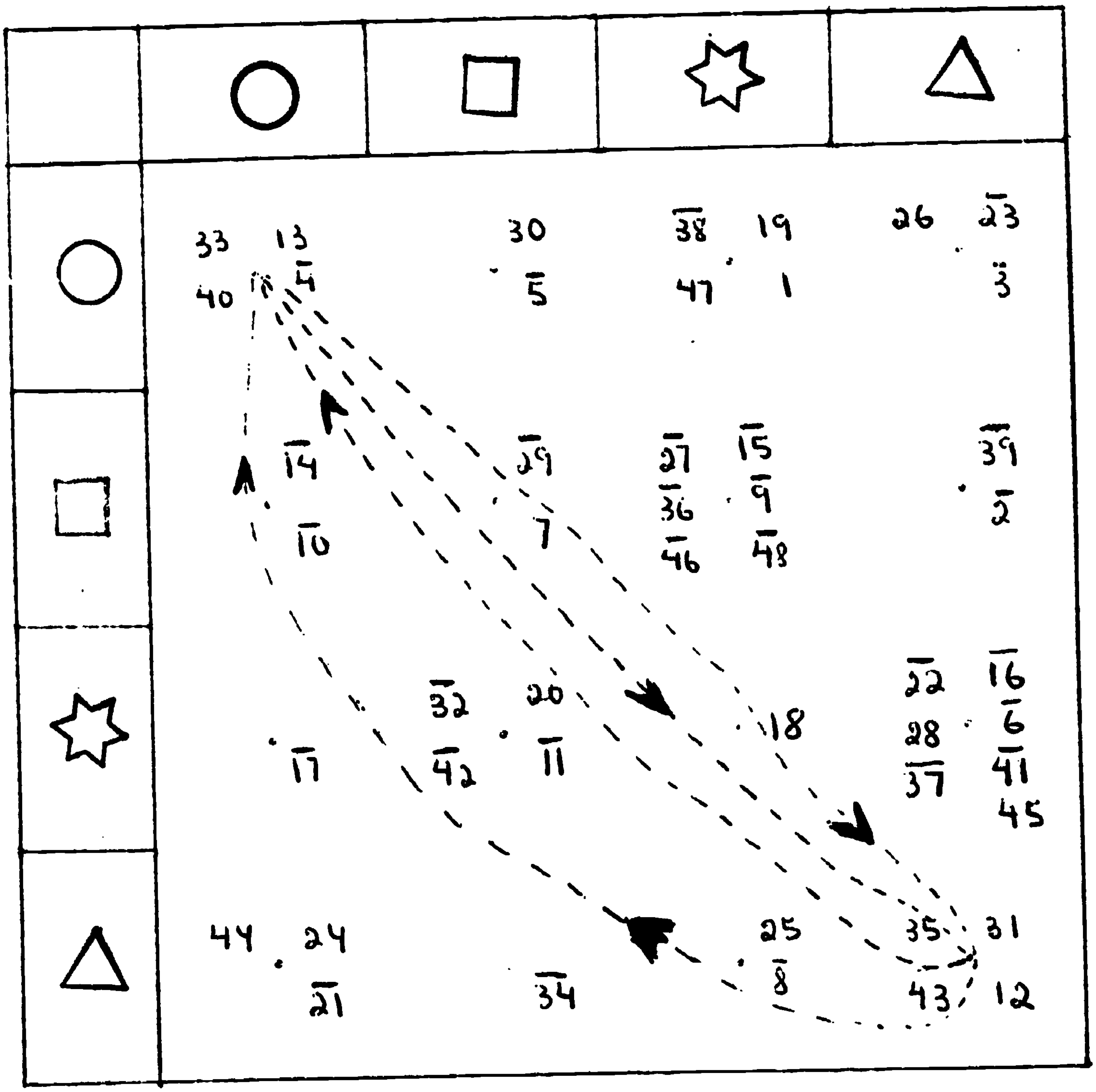
14
31 $\overline{19}$
$\overline{38}$ $\overline{1}$
$\overline{43}$
$\overline{30}$ 17
4
$\overline{42}$ $\overline{34}$
$\overline{9}$
32 20
16
$\overline{26}$
$\overline{40}$ $\overline{24}$
3
47 29
10

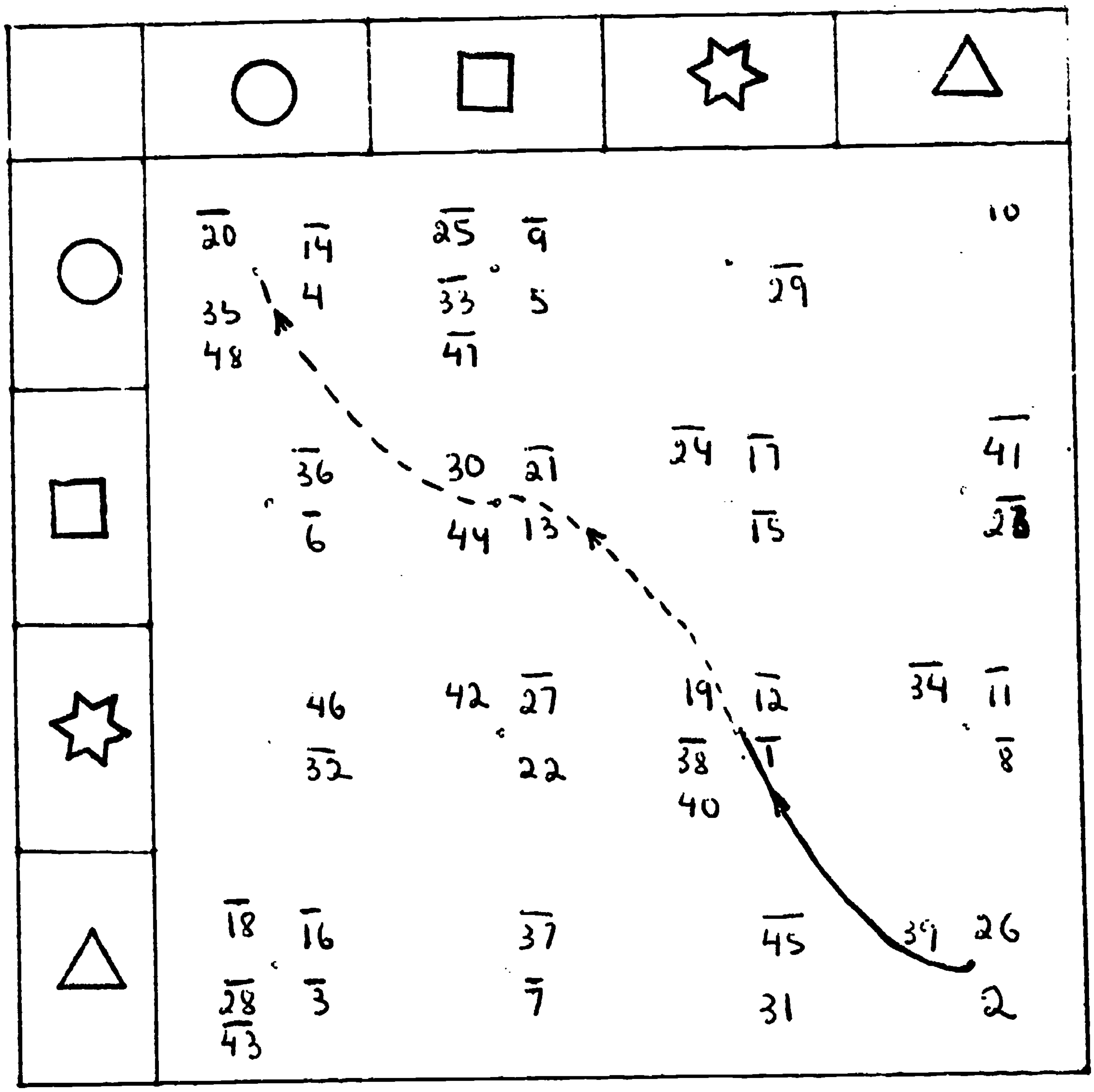
0 2 2

GRAPH II


33 13
40 18 4
30
5
38 19
47 1
26 23
3
14
10
29
7
27 15
36 9
46 48
39
2
17
32 20
42 11
18
22 16
28 6
37 41
45
44 24
21
34
25
8
35 31
43 12

0 3 1

GRAPH III



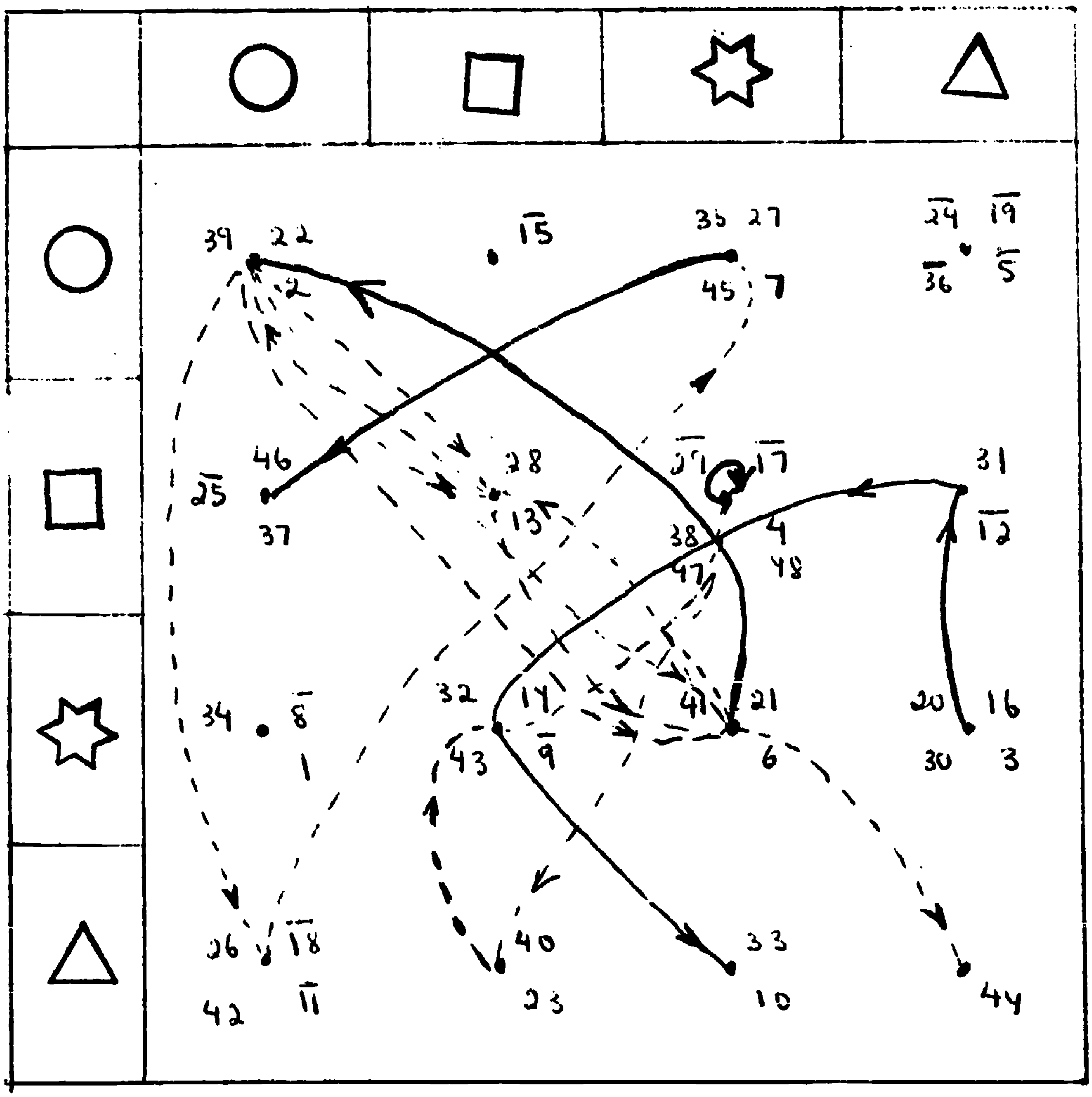
044

GRAPH IV

	○	□	✡	△
○	33 15 $\overline{40}$ $\overline{10}$	$\overline{31}$ $\overline{21}$ $\overline{6}$	$\overline{36}$ $\overline{23}$ $\overline{9}$	43 37 $\overline{22}$
□	$\overline{42}$ $\overline{38}$ 24 44	25 $\overline{4}$	$\overline{41}$ $\overline{13}$	$\overline{30}$ $\overline{18}$
✡	29 $\overline{20}$ $\overline{7}$	$\overline{16}$ 2 $\overline{34}$ 1	28 $\overline{19}$ $\overline{46}$ $\overline{11}$ 48	$\overline{39}$ $\overline{35}$
△	14 $\overline{3}$	$\overline{45}$ $\overline{26}$ $\overline{47}$ $\overline{12}$	27 $\overline{8}$	32 $\overline{17}$ $\overline{5}$

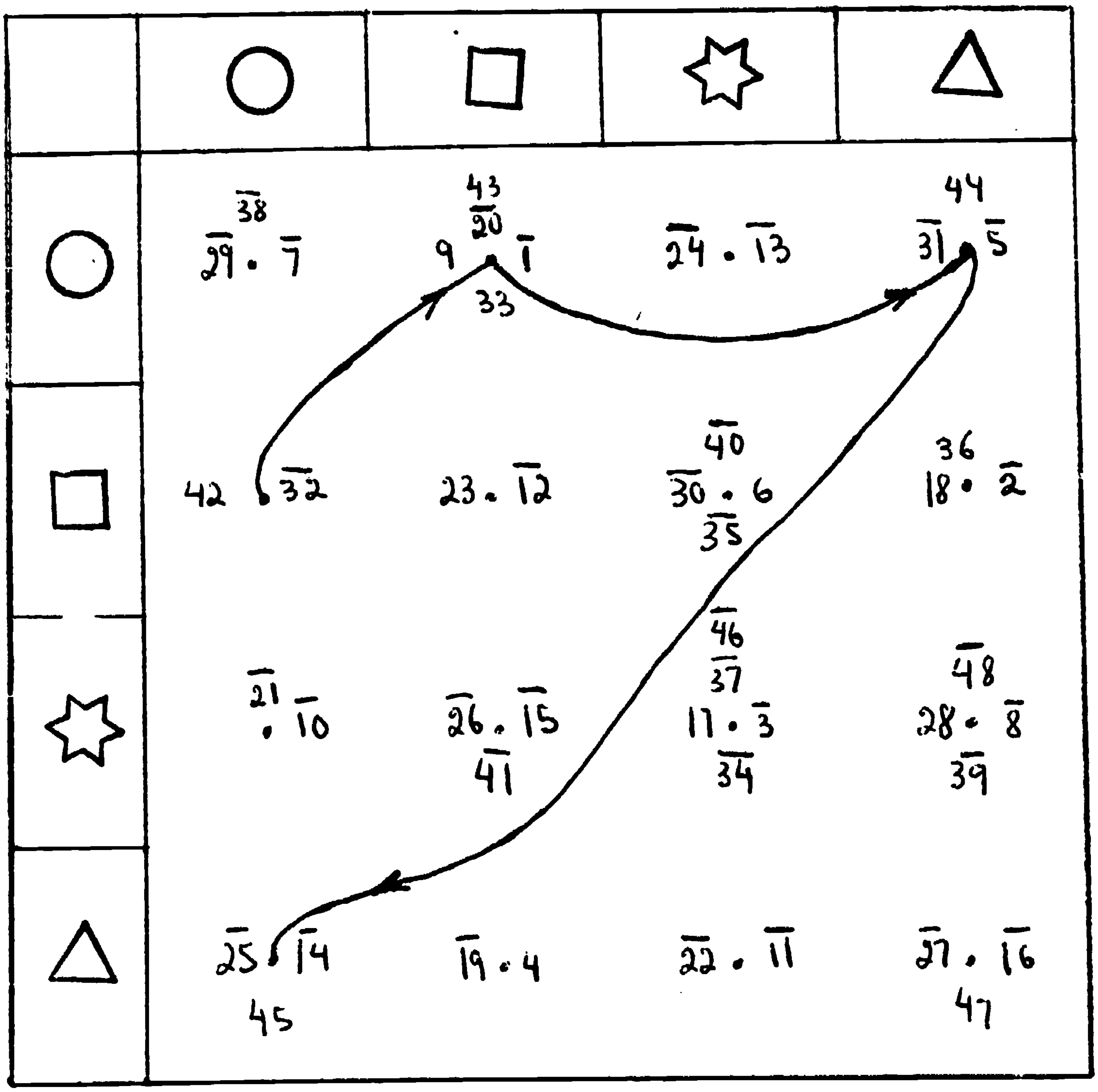
1	1	4

GRAPH V



1 3 3

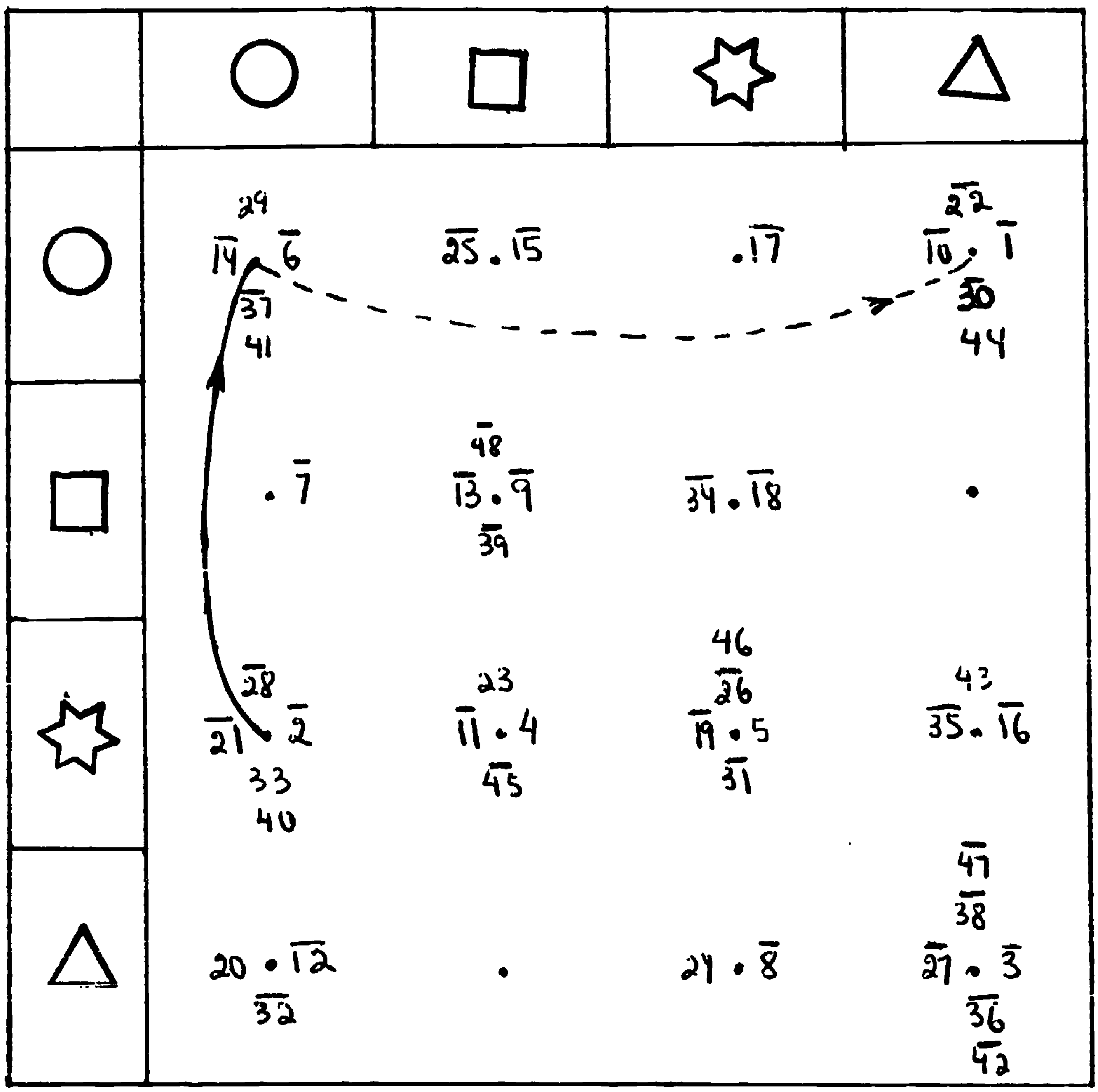
GRAPH VI



38
29 • 7
43
20
9 • 1
33
24 • 13
44
31 • 5
42 • 32
23 • 12
40
30 • 6
35
36
18 • 2
21
• 10
26 • 15
41
46
37
11 • 3
34
48
28 • 8
39
25 • 14
45
19 • 4
22 • 11
27 • 16
47

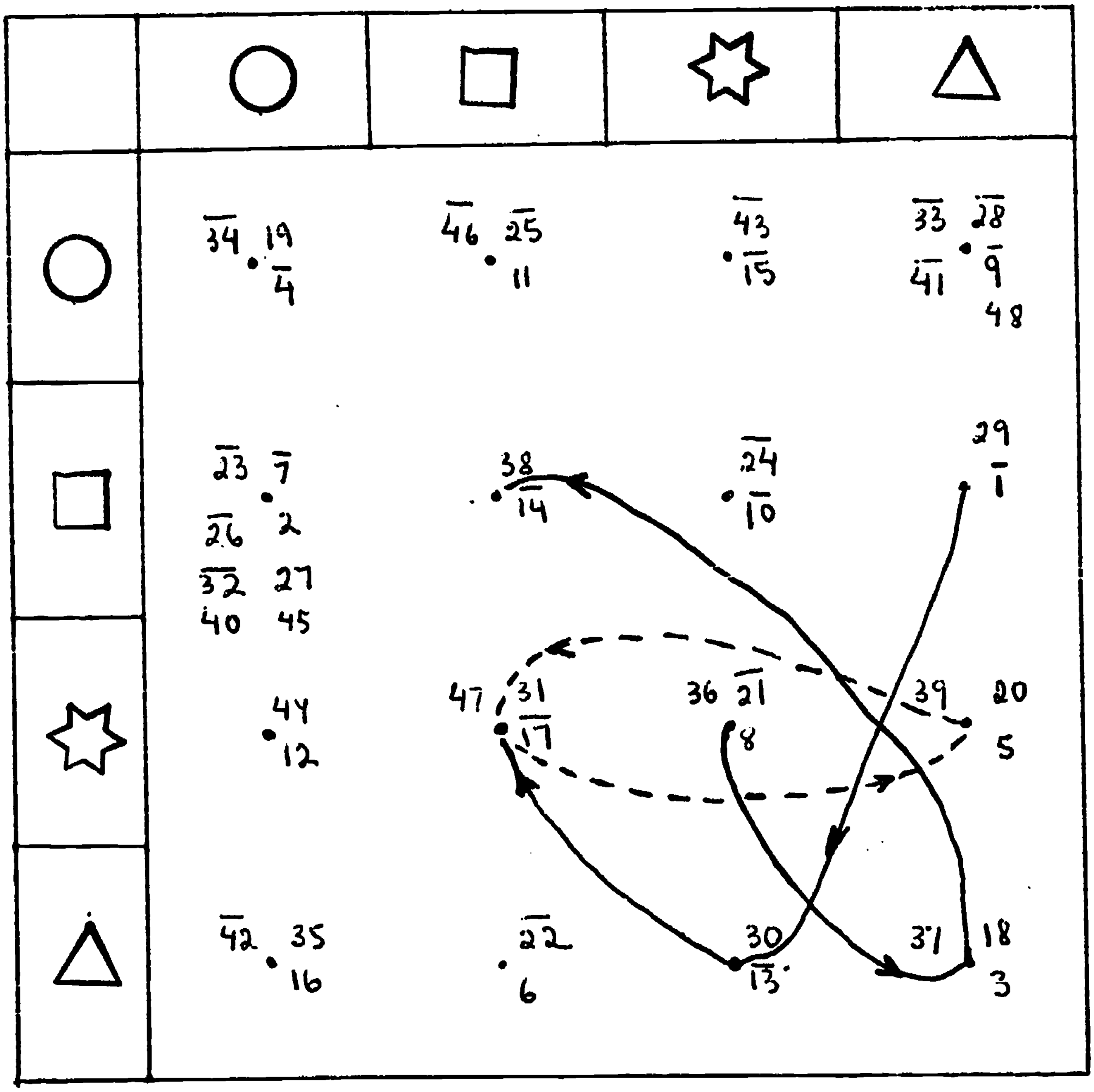
141

GRAPH VII



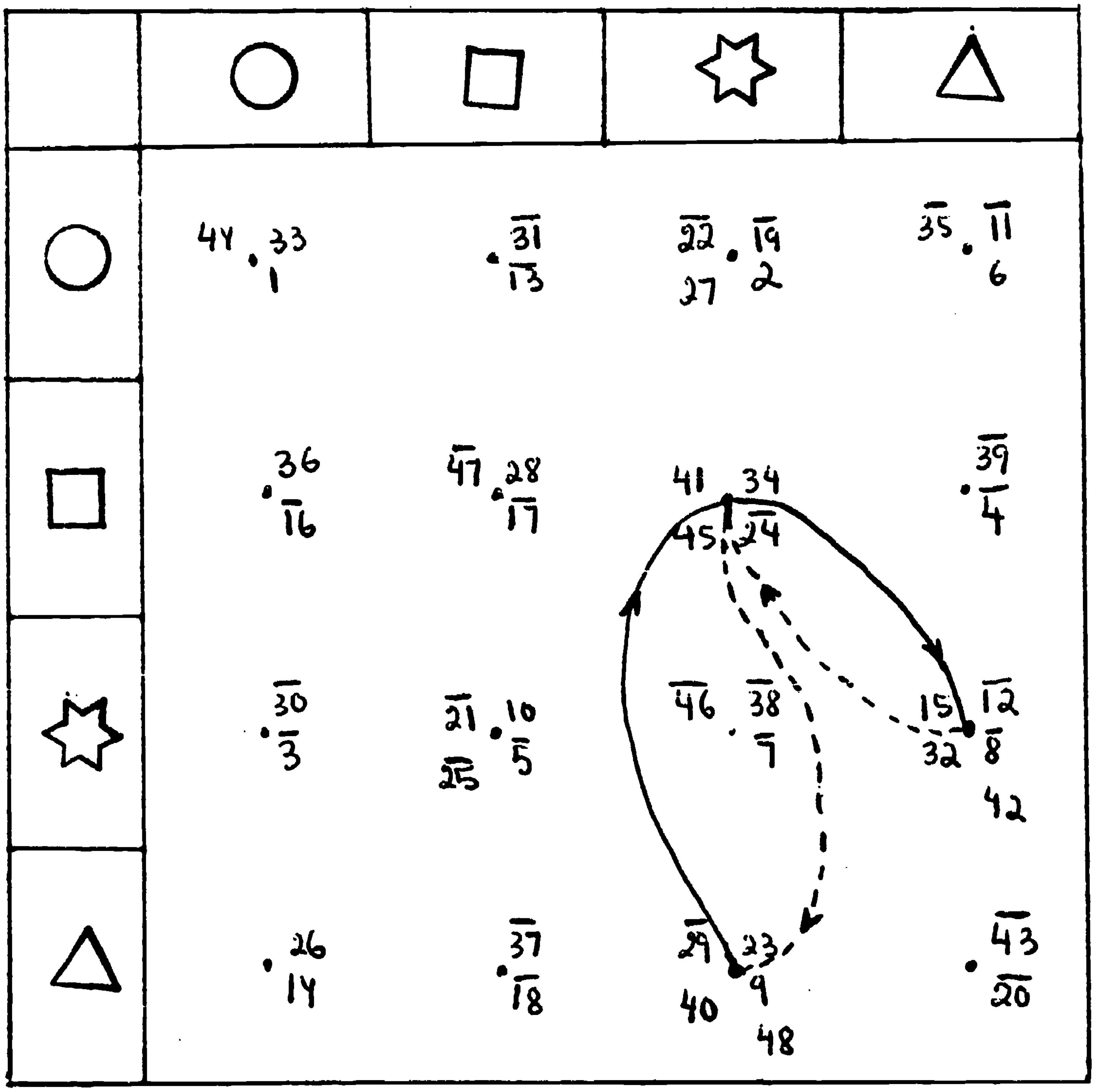
1 6 2

GRAPH VIII.



1 9 4

GRAPH IX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26 14
37 18
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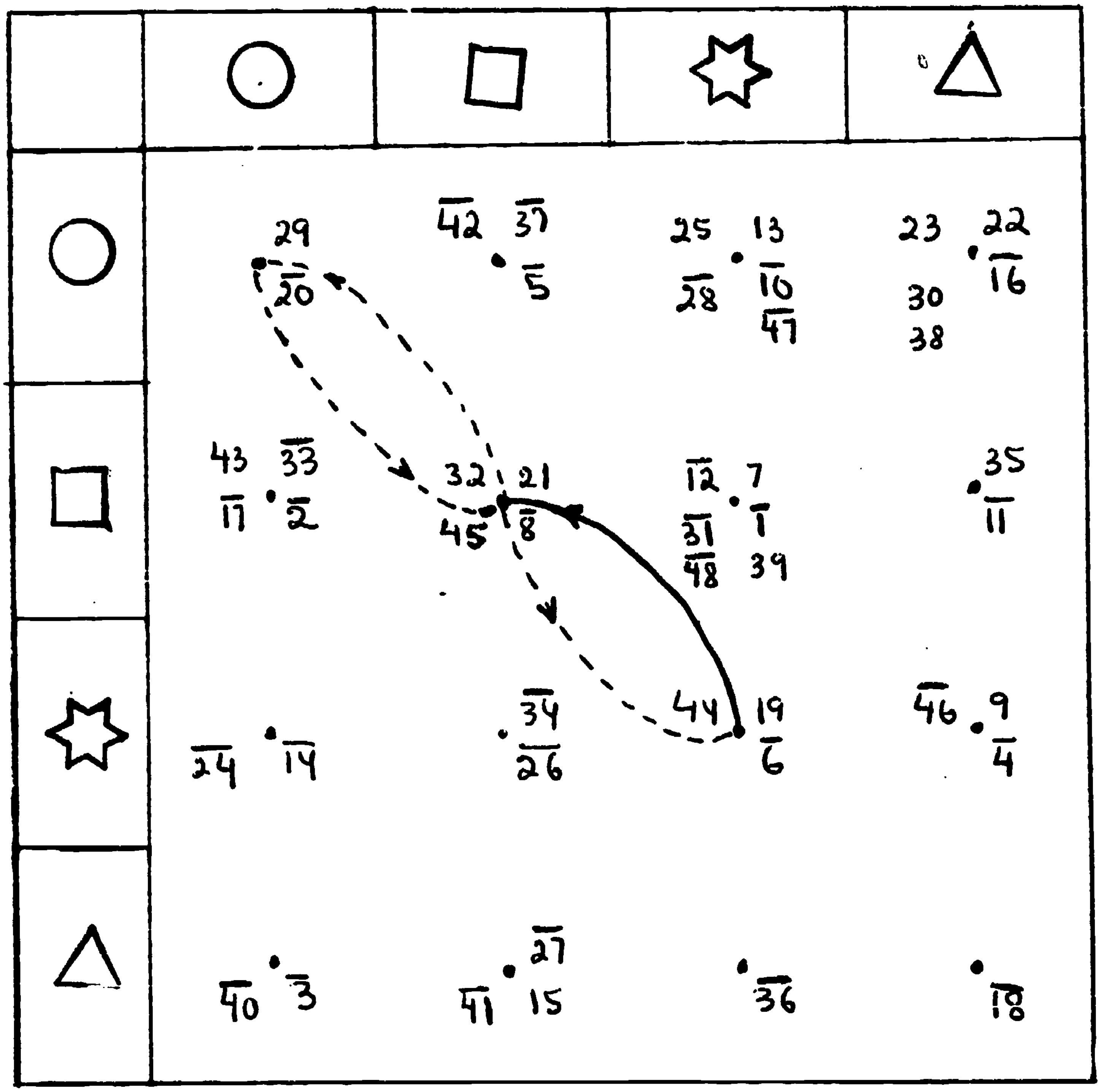
43 20

214

GRAPH X



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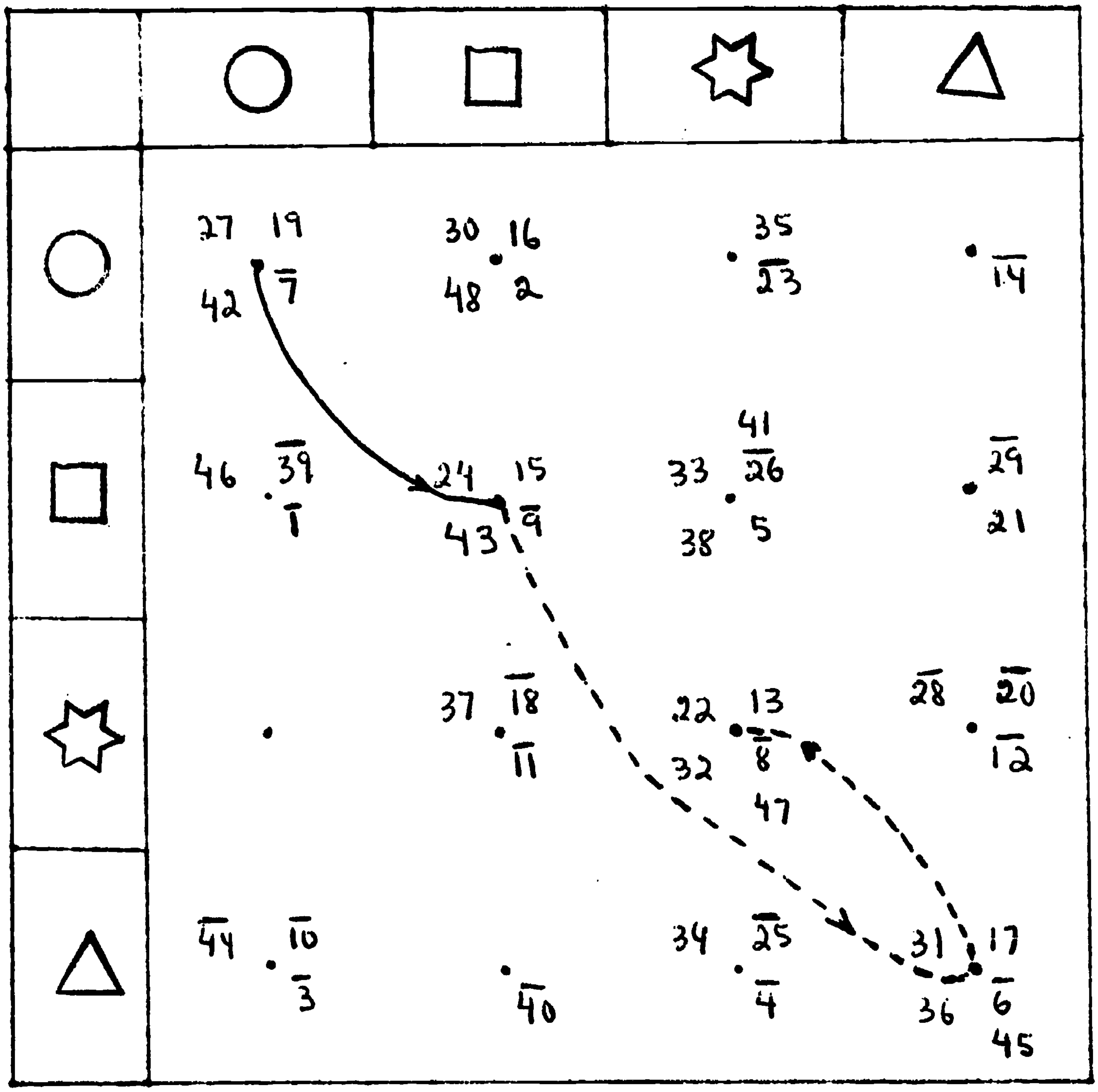
GRAPH XI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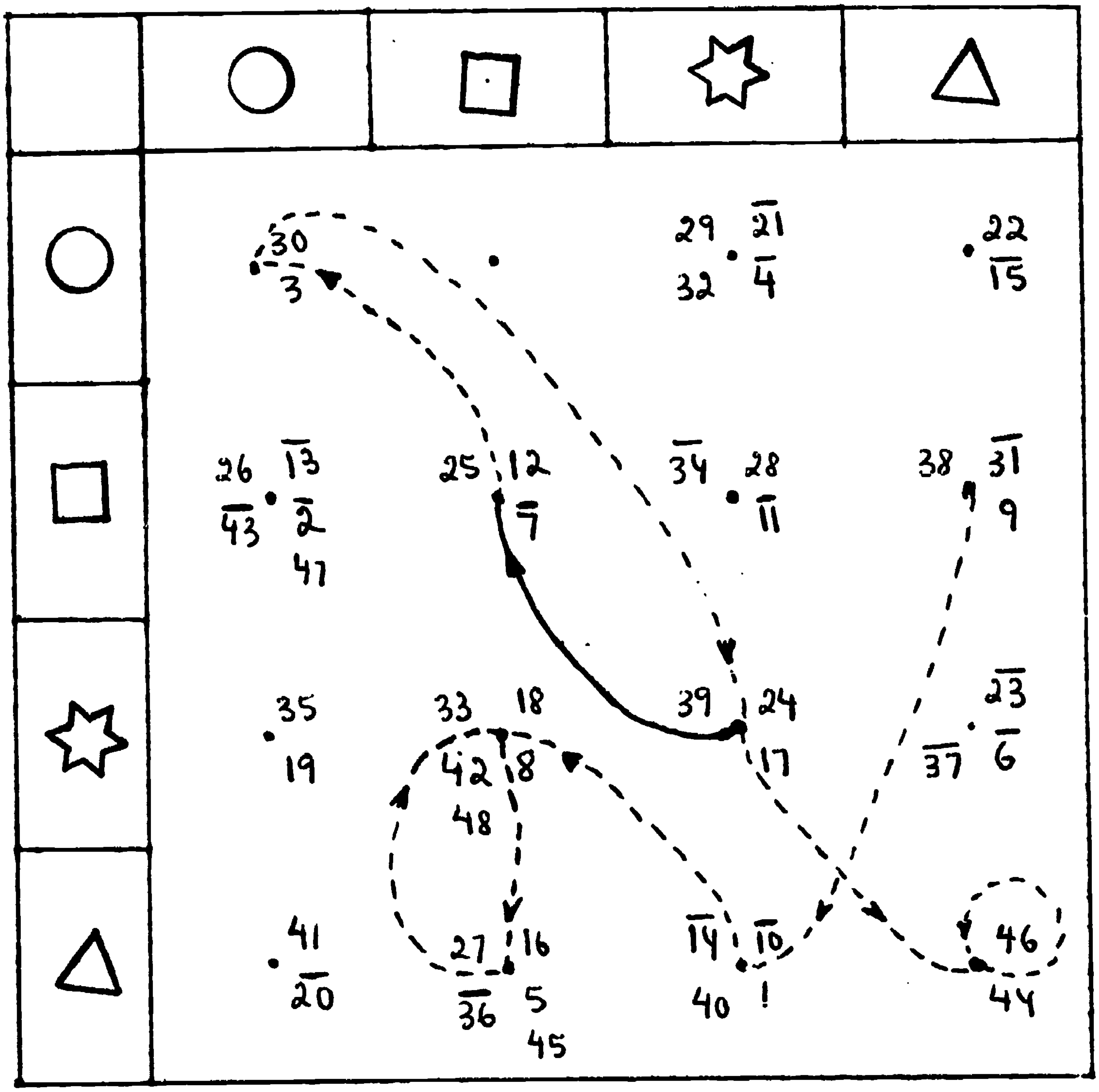
45

2 6 2

GRAPH XII



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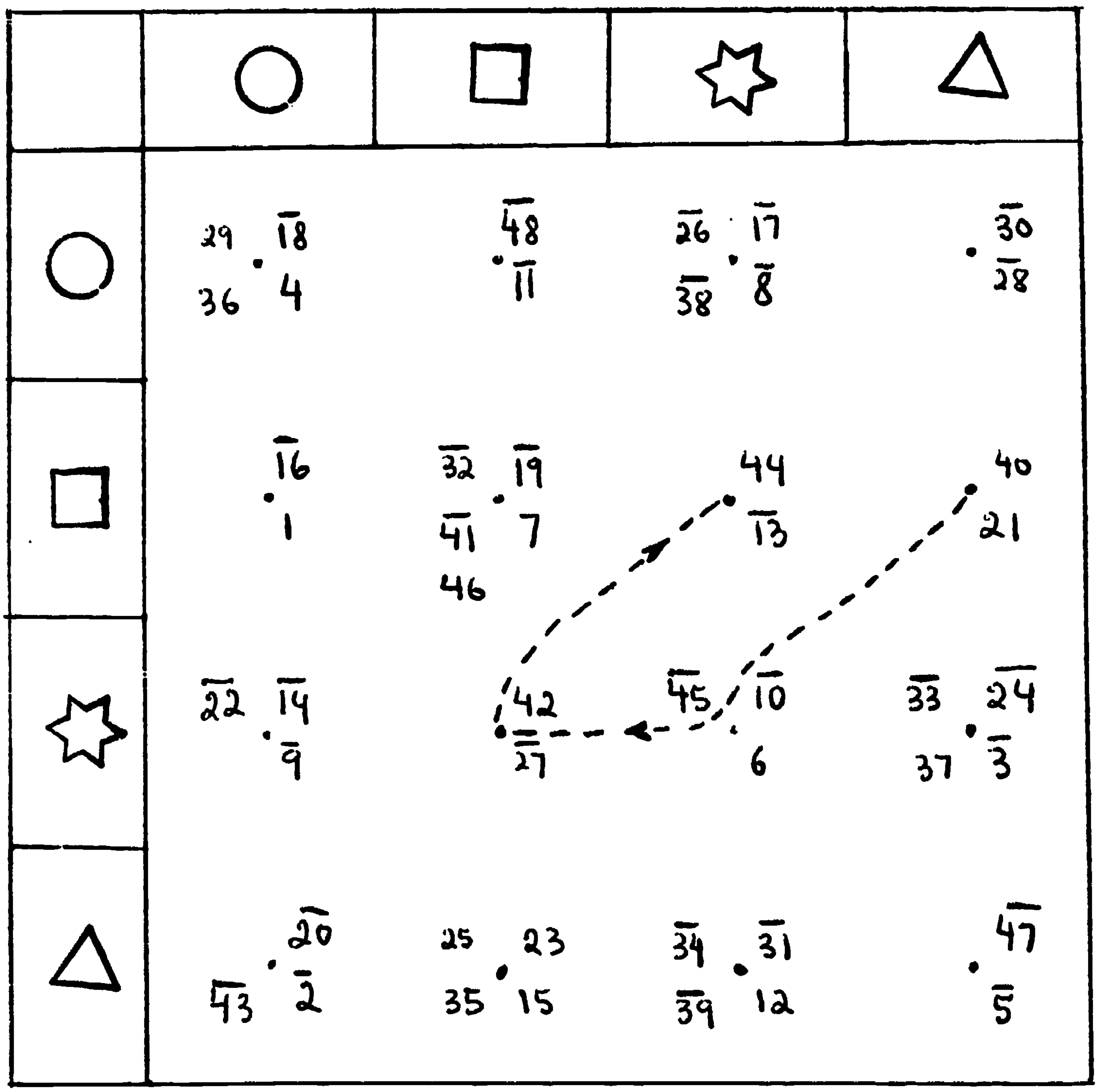
GRAPH XIII



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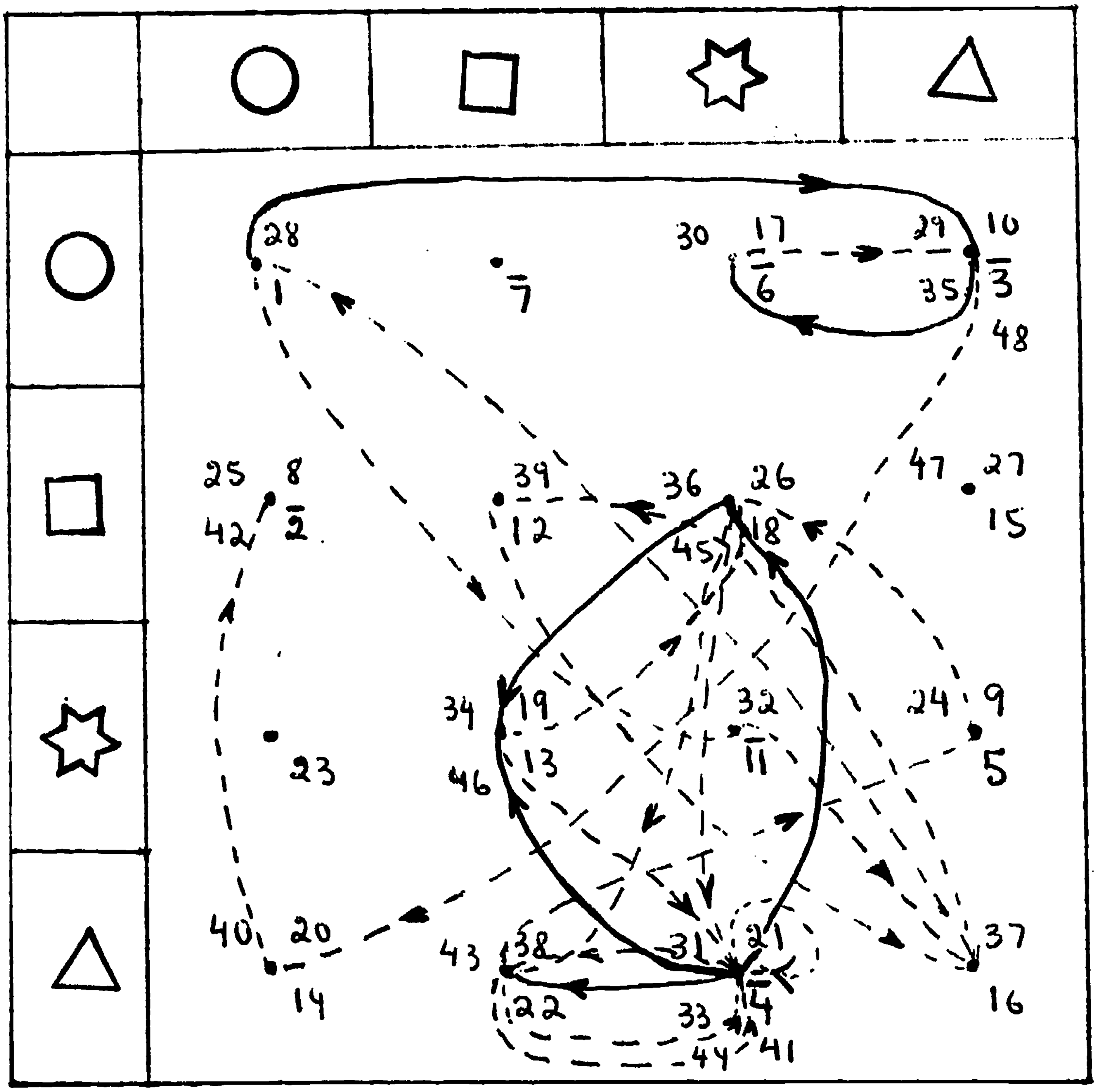
3 1 2

GRAPH XIV



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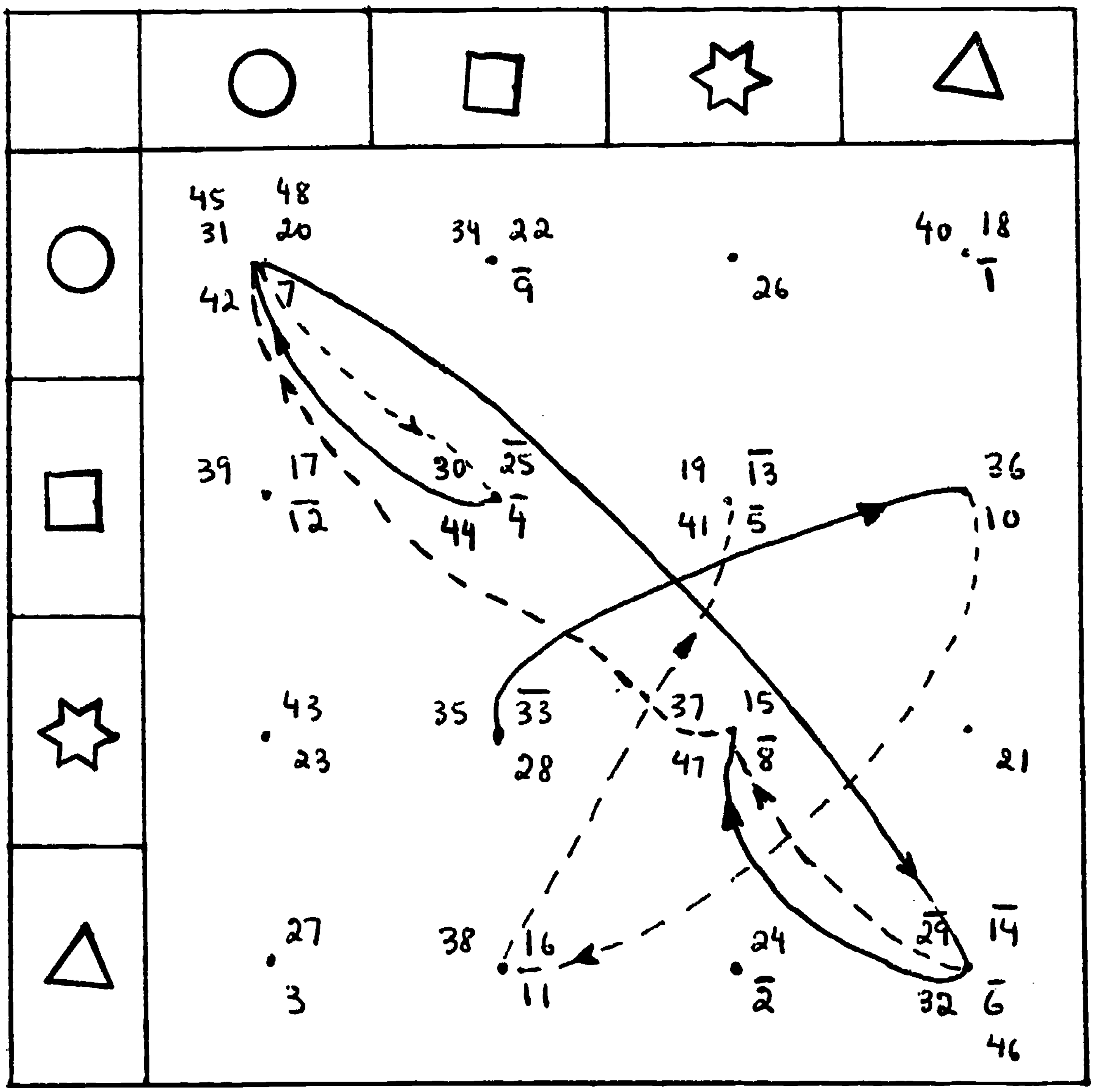
GRAPH XV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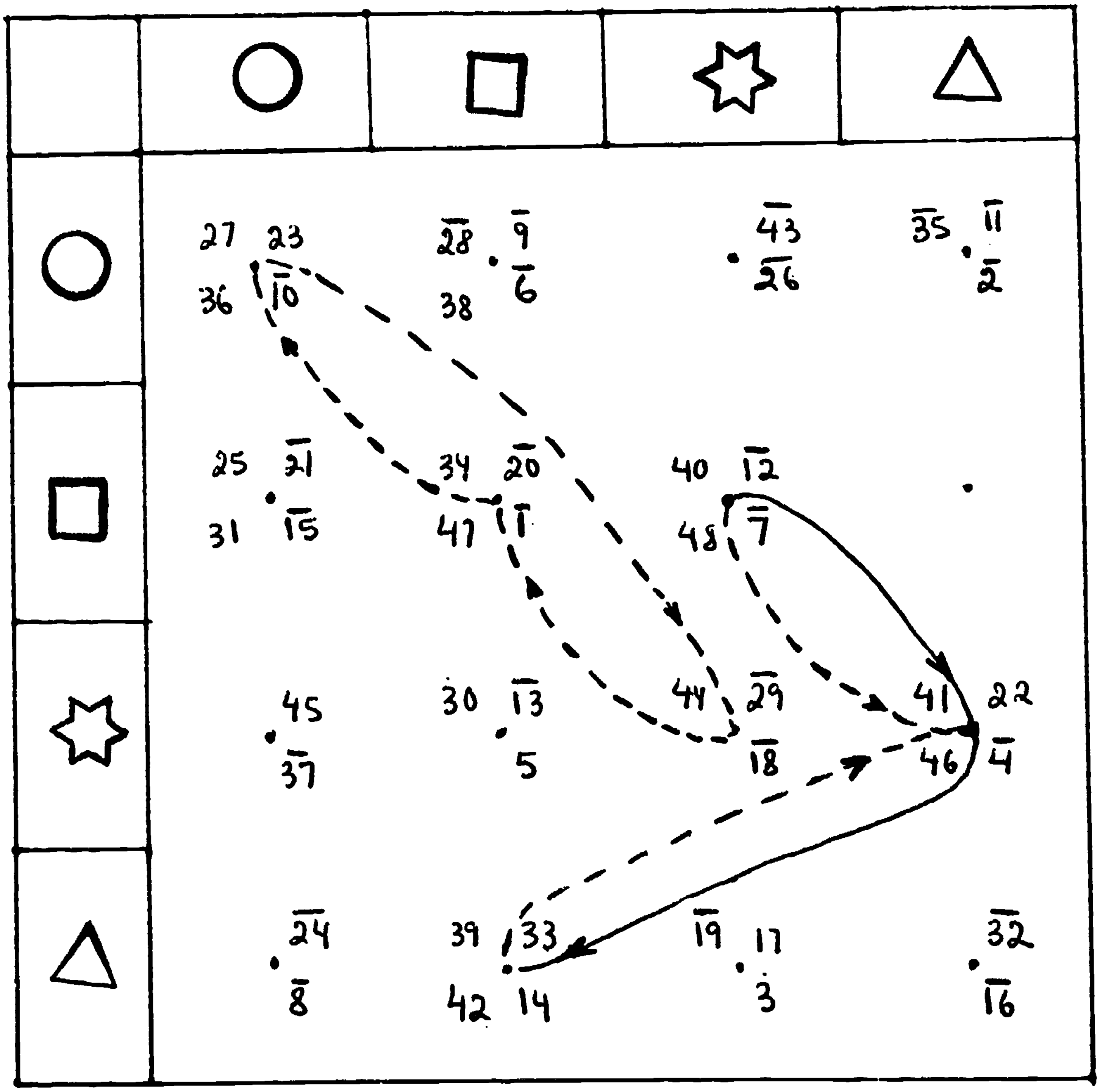
342

GRAPH XVI



353

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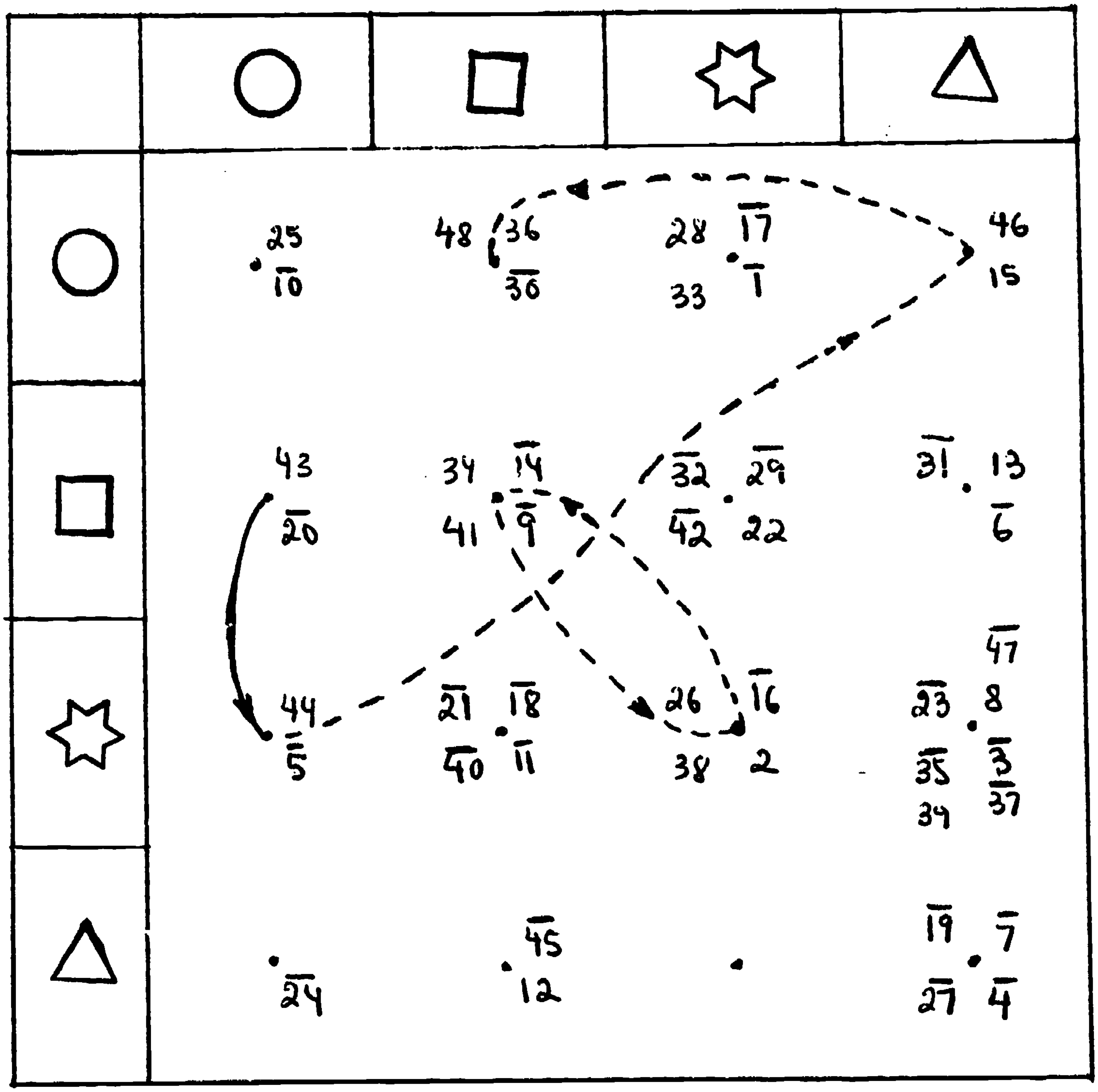

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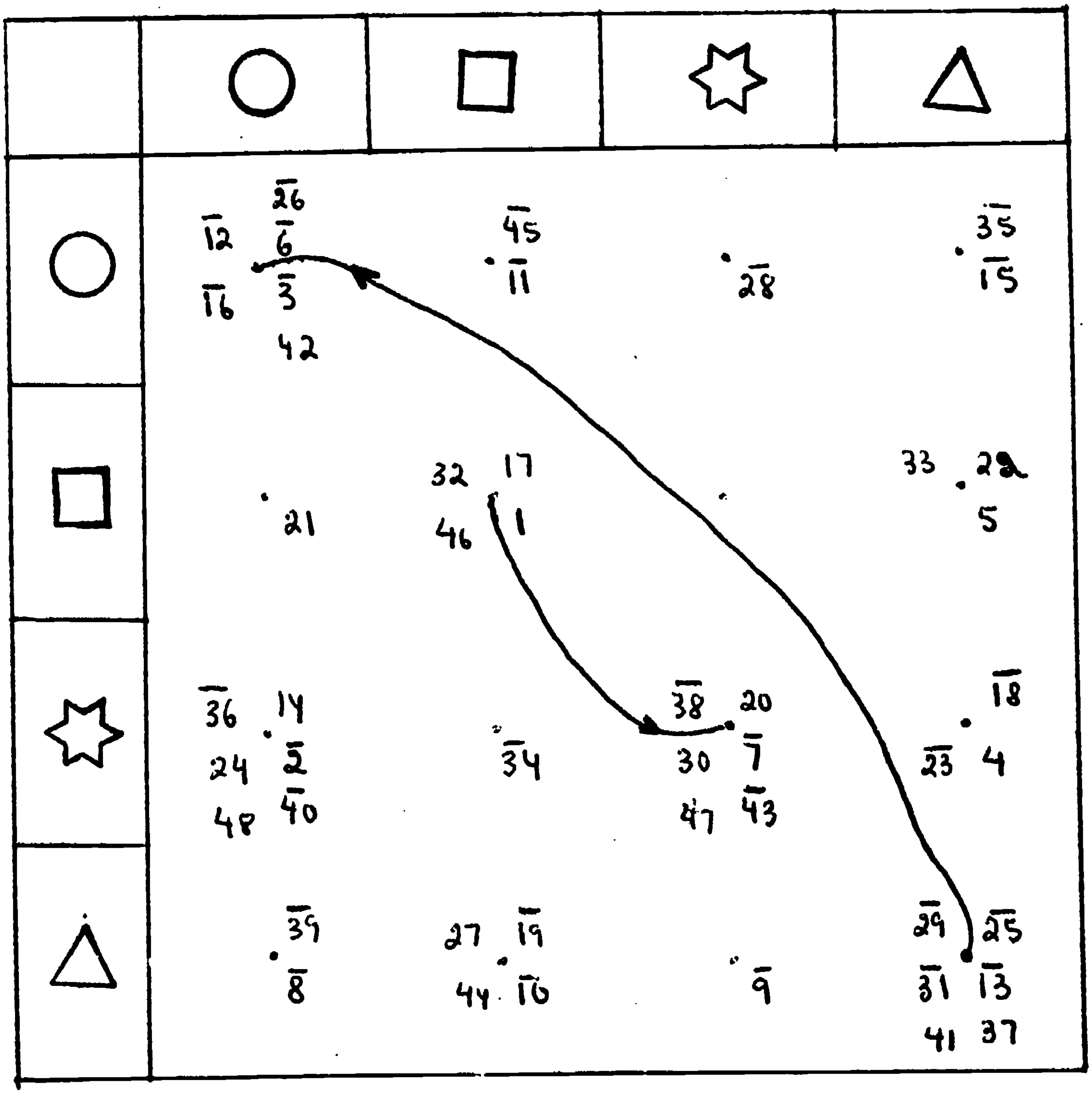
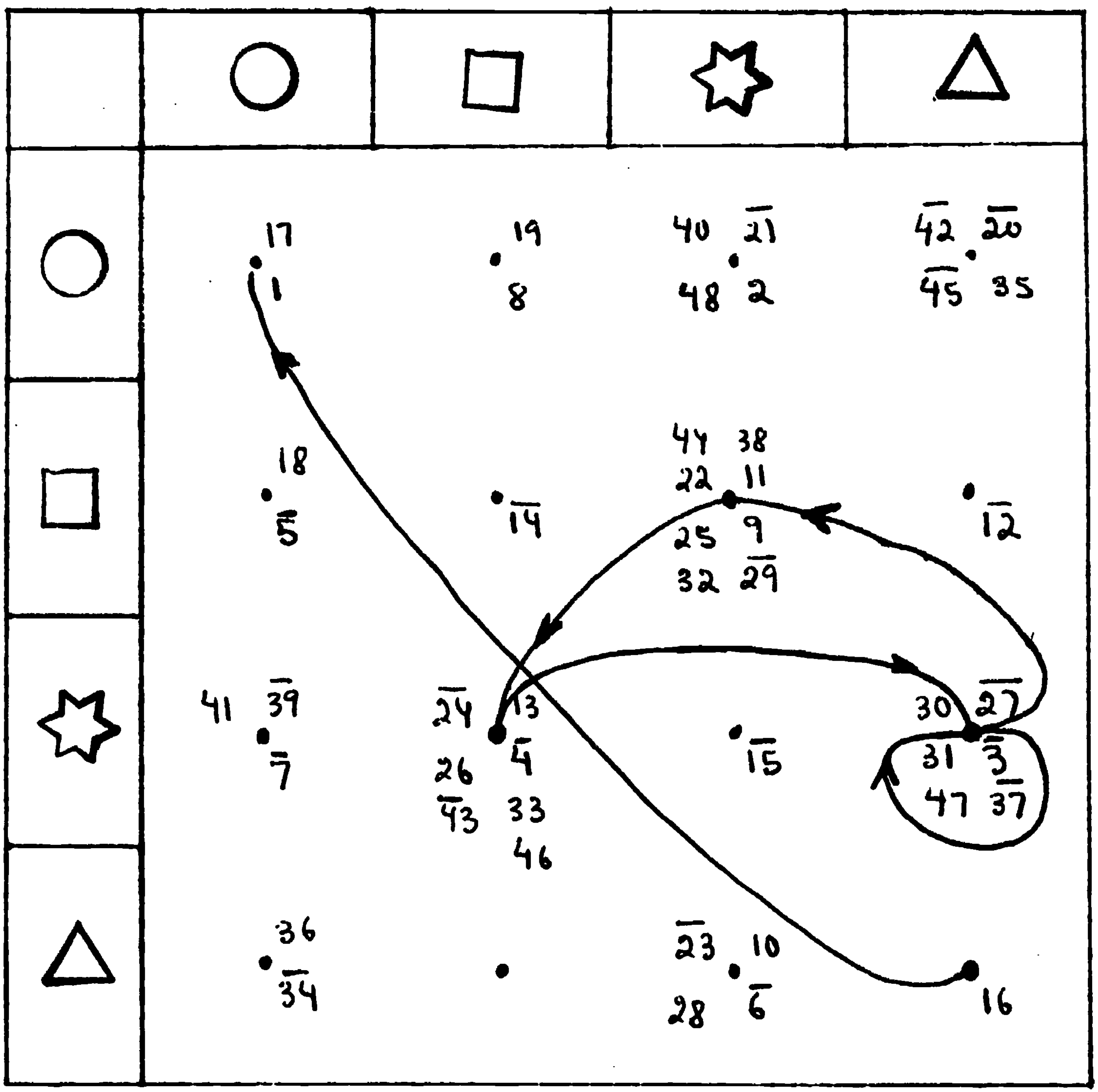


| 3 | 8 | 3 |

GRAPH XIX



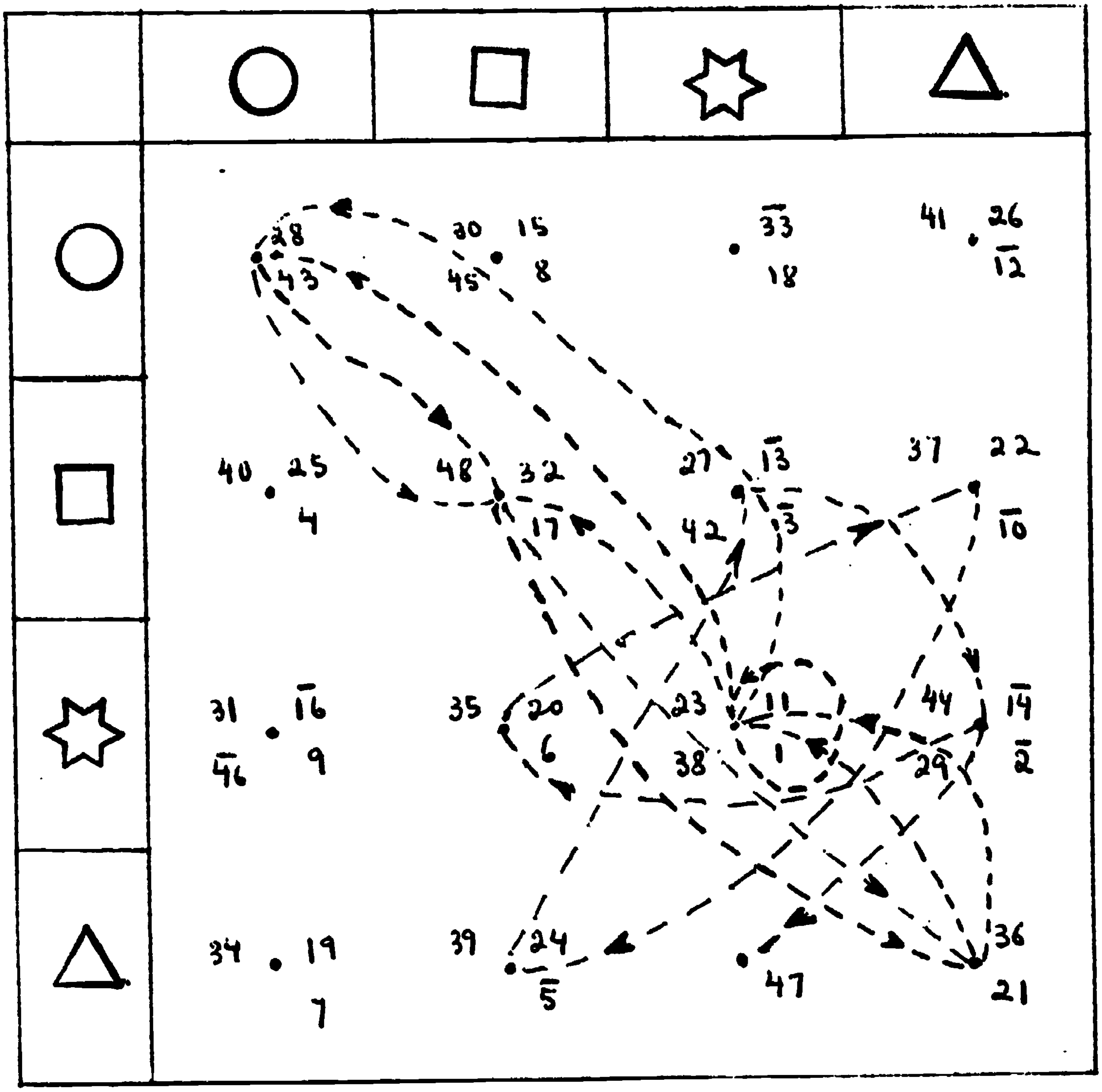
3 9 4

GRAPH XX

	○	□	☆	△
○	38 45 19 13 $\overline{31}$ $\overline{7}$	48 25 8 36 3		$\overline{44}$ $\overline{32}$ 15 $\overline{37}$ 1
□	$\overline{12}$ $\overline{9}$	$\overline{40}$ $\overline{28}$ $\overline{17}$	41 23 20 27 $\overline{16}$	43 $\overline{39}$ 46 $\overline{21}$
☆	$\overline{30}$ 14	$\overline{2}$	$\overline{34}$ $\overline{26}$ $\overline{6}$	33 10 $\overline{4}$
△		$\overline{22}$	29 18 $\overline{5}$	47 $\overline{35}$ 24 $\overline{42}$ $\overline{11}$

402

GRAPH XXI



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